A study of the gravitational wave pulsar signal with orbital and spindown effects[1]

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Abstract: In this work, we present an analytic and a preliminary numerical analysis of the gravitational wave signal from a pulsar that includes simple spindown effects. We estimate the phase corrections to a monochromatic source signal due to rotational and elliptical orbital motion of the Earth, and perturbations due to Jupiter and the Moon. We briefly discuss the Fourier transform of such a signal, expressed in terms of well-known special functions, and its applications.

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Résumé : Nous présentons ici une analyse analyrique et une nouvelle analyse numérique d'un signal d'onde gravitationnelle d'un pulsar, incluant la décroissance rotationnelle. Nous estimons les corrections de phase du signal monochromatiques dues aux mouvements rotationnel et orbital elliptique de la Terre, ainsi qu'aux perturbations par Jupiter et la Lune. Nous discutons brièvement la transformée de Fourier d'un tel signal, exprimée en terme de fonctions bien connues et ses applications.

[Traduit par la Rédaction]

1. Introduction

The detection of gravitational waves (GW) from astrophysical sources is one of the outstanding problems in experimental gravitation today. Gravitational wave detectors like the LIGO, VIRGO, LISA, TAMA 300, GEO 600, and AIGO are opening a new window for the study of a great variety of nonlinear curvature phenomena. Detection of GW necessitates sufficiently long observation periods to attain an adequate signal-to-noise ratio. The data analysis for continuous GW, for example, from rapidly spinning neutron stars, is an important problem for ground-based detectors that demands analytic, computational, and experimental ingenuity.

In recent works [1, 2], we have implemented the Fourier transform (FT) of the Doppler-shifted GW signal from a pulsar with the plane wave expansion in spherical harmonics (PWESH). It turns out that the consequent analysis of the FT of the GW signal from a pulsar has a very interesting

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and convenient development in terms of the resulting spherical Bessel, generalized hypergeometric, Gamma and Legendre functions. These works considered the frequency modulation of a GW signal due to rotational and circular orbital motions of the detector on the Earth. The formalism has a nice analytic representation for the GW signal in terms of the special functions above. The signal can then be studied as a function of various parameters associated with the GW pulsar signal, as well as the orbital and rotational parameters of the Earth. An analytic study of the perturbation effects has been done recently [3]. The numerical analysis of this analytical expression for the signal offers a challenge for efficient and fast numerical and parallel computation.

In the present paper, such a numerical analysis due to rotational and orbital eccentric motions of the Earth, perturbations due to Jupiter and the Moon, and approximate pulsar spindown effects is considered. In this work, we present a variation of a parametrized model of pulsar spindown discussed by Brady and co-workers [4] and Jaranowski and co-workers. [5] for the GW frequency and the phase measured at the ground-based detector that includes the spindown parameters. The various types of motion of the detector can be naturally incorporated into this simple parametrized spindown model. Effects of amplitude modulation (AM), though not considered in this paper, can also be incorporated into this formalism without great difficulty. The numerical study outlined in this paper may be relevant to GW detectors.

2. Gravitational-wave signal with spindown corrections

A slightly modified form of the parameterization for the phase of a gravitational wave signal given by Brady and co-workers [4] is given by

$$\Phi(t,\lambda) = \pi f_0 \left(x + \sum_{k=0}^{\infty} \frac{s_k}{k+1} x^{k+1} \right) \tag{1}$$

($x = t + (\boldsymbol{r}/c) \cdot \hat{n}$, where $\boldsymbol{r}(t)$ is the position vector of the detector in the solar system barycentre frame, $\hat{n}$ is a unit vector in the direction of the GW source, and c is the speed of light; f_0 = source GW frequency.) The s_k are spindown parameters, with $-\tau_{\text{min}}^{-k} \leq s_k \leq \tau_{\text{min}}^{-k}$. ($\tau_{\text{min}}$ is a parameter specifying the spindown age of the pulsar.) We let $s_k = T_k(z)/\tau_{\text{min}}^{k}$, where $T_k(z)$ are the Chebyshev polynomials, with $-1 \leq T_k(z) \leq 1$, and z is a control parameter with domain $-1 \leq z \leq 1$, which together allow s_k to be adjusted. Equation (1) becomes

$$\Phi(t,\lambda) = \pi f_0 \left[x + \tau_{\text{min}} \sum_{k=0}^{\infty} \frac{T_k(z)}{k+1} \left(\frac{x}{\tau_{\text{min}}} \right)^{k+1} \right] \tag{2}$$

The advantage of representing s_k using Chebyshev polynomials is that the summation in (2) can now be evaluated explicitly [6]

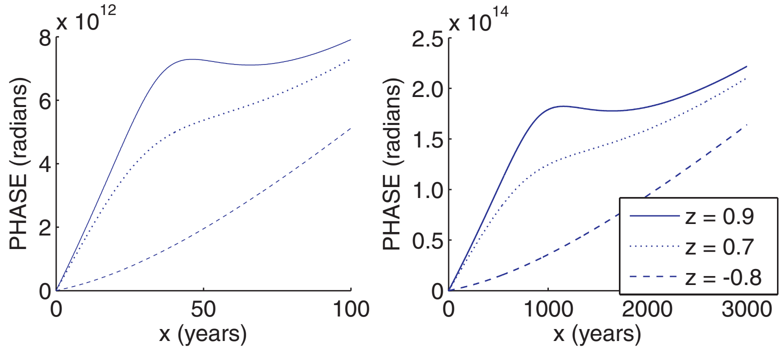
$$\sum_{k=0}^{\infty} \frac{T_k(z)}{k+1} \left(\frac{x}{\tau_{\text{min}}} \right)^{k+1} = -\frac{1}{2} \ln \left(1 - 2 \left(\frac{x}{\tau_{\text{min}}} \right) z + \left(\frac{x}{\tau_{\text{min}}} \right)^2 \right) \tag{3}$$

The numerical estimates enable a plot of the phase (2) as a function of x with $\tau_{\text{min}} = 40$ and 1000 years for $z = -0.8$, 0.7, and 0.9, with $f_0 = 1$ kHz (see Fig. 1).

The GW signal is now given by

$$\mathrm{e}^{i\Phi(t,\lambda)} = \mathrm{e}^{i\pi f_0 x} \left[1 - \frac{i\pi f_0 \tau_{\text{min}}}{2} \left(-2 \left(\frac{x}{\tau_{\text{min}}} \right) z + \left(\frac{x}{\tau_{\text{min}}} \right)^2 \right) + \ldots \right] \tag{4}$$

Fig. 1. Plot of the phase of gravitational-wave signal as a function of x for $\tau_{\min} = 40$ years (left) and $\tau_{\min} = 1000$ years (right) for different values of z.



We have used the binomial expansion inside the square brackets. We define the spindown moment integrals as follows:

$$\int_0^T \mathrm{e}^{i\pi f_0 x} \cdot \mathrm{e}^{-i2\pi f t} \cdot \left(\frac{x}{\tau_{\min}}\right)^k \mathrm{d}t \tag{5}$$

Note that each term of the signal (4), after Fourier transformation, can be represented by such a spindown moment integral. If we define

$$I_{\text{generic}} = \int_0^T \mathrm{e}^{(i\pi f_0 x - i2\pi f t)}\, \mathrm{d}t \tag{6}$$

then the kth spindown moment integral can be written as

$$\frac{1}{(i\pi\tau_{\min})^k}\frac{\partial^k I_{\text{generic}}}{\partial f_0^k} \tag{7}$$

Thus the Fourier transform of the signal can be written in terms of the partial derivatives with respect to f_0 of I_{generic}, which can be evaluated analytically. We emphasize that this spindown model is heuristic. A detailed numerical analysis with spindown effects in a more generic model is currently under study.

3. Corrections to the position vector of the detector

In this section, we briefly outline the important phase corrections to account for the Keplerian ellipse, Earth's rotation, and the approximate perturbations due to Jupiter and the Moon, for the position vector $\boldsymbol{r}(t)$ of the detector.

An expansion of the elliptical orbit of the Earth, to second-order in eccentricity, leads to the following expressions for the x and y components of $\boldsymbol{R}_{\text{orb}}$, the vector specifying the Earth's orbital position (the z component is 0) [7]:

$$x(t) = a\left(1 - \frac{3}{8}e^2\right)\cos M + \frac{ae}{2}\cos 2M + \frac{3}{8}ae^2\cos 3M - \frac{3}{2}ae \tag{8a}$$

$$y(t) = a\left(1 - \frac{3}{8}e^2\right)\sin M + \frac{ae}{2}\sin 2M + \frac{3}{8}ae^2\sin 3M - \frac{1}{4}ae^2\sin M \tag{8b}$$

(a = Sun–Earth distance, $M = \omega_{\text{orb}}t$, and e = eccentricity of orbit.) If the radius of Jupiter's orbit is much greater than that of the Earth, we can consider the Sun, to a first approximation, as moving around the Sun–Jupiter (S-J) barycentre, and the Earth orbiting the Sun. Then, the perturbation $\boldsymbol{R}_{\mathrm{J}}$ due to Jupiter is [3]

$$\boldsymbol{R}_{\mathrm{J}} = [R_{\mathrm{J}}\,(\cos(\omega_{\mathrm{J}}t) - 1)\,,\, R_{\mathrm{J}}\sin(\omega_{\mathrm{J}}t)\,,\, 0] \tag{9}$$

where ω_{J} is the orbital angular frequency of Jupiter, and R_{J} is the distance of the Sun from the S-J barycentre.

Similarly, the vector from the Earth–Moon barycentre to the Earth is

$$\boldsymbol{R}_{\mathrm{M}} = \begin{bmatrix} R_{\mathrm{EM}}\,(\cos(\omega_{\mathrm{M}}t) - 1) \\ R_{\mathrm{EM}}\sin(\omega_{\mathrm{M}}t) \\ 0 \end{bmatrix} \tag{10}$$

Here ω_{M} is the Moon's sidereal orbital angular frequency, and R_{EM} is the distance from Earth to the Earth–Moon barycentre. Also, the vector specifying the detector's position due to the rotating Earth, $\boldsymbol{R}_{\mathrm{rot}}$, is given by

$$\boldsymbol{R}_{\mathrm{rot}} = \begin{bmatrix} R_{\mathrm{E}}\sin\alpha\,(\cos(\omega_{\mathrm{rot}}t) - 1) \\ R_{\mathrm{E}}\sin\alpha\sin(\omega_{\mathrm{rot}}t)\cos\varepsilon \\ R_{\mathrm{E}}\sin\alpha\sin((\omega_{\mathrm{rot}}t)\sin\varepsilon \end{bmatrix} \tag{11}$$

where R_{E} = Earth's radius, α = detector co-latitude, ε = angular tilt of Earth's axis, and ω_{rot} = Earth's sidereal rotational angular frequency. The position of the detector, which also now includes simple spindown effects, is now given in the form

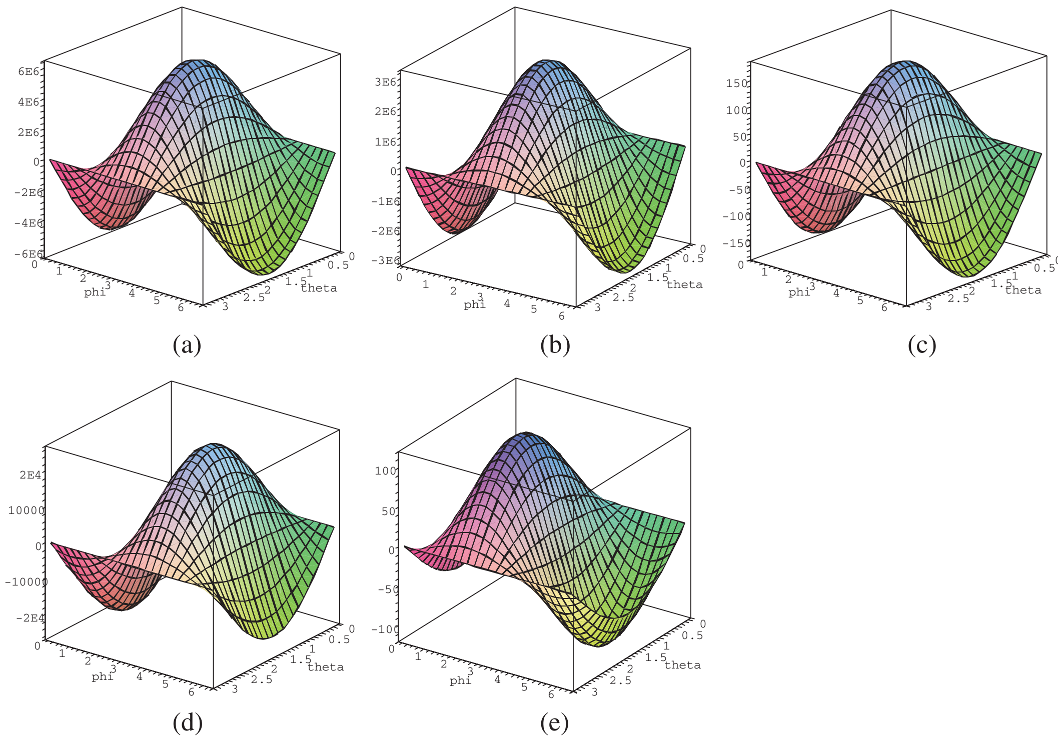
$$\boldsymbol{r}(t) = \boldsymbol{R}_{\mathrm{orb}}(t) + \boldsymbol{R}_{\mathrm{rot}}(t) + \boldsymbol{R}_{\mathrm{J}}(t) + \boldsymbol{R}_{\mathrm{M}}(t) \tag{12}$$

4. Numerical estimates of the phase due to perturbations

The contribution of a perturbation to the phase of the GW signal (in radians), crucial in determining search templates for the GW forms, is given by $\Phi(t) = 2\pi f_0(\boldsymbol{R}_i(t)\cdot\hat{n}/c)$, where $\boldsymbol{R}_i(t)$ is the contributing vector for the perturbation [2]. Our numerical estimates of the phase contributions for the Earth's circular and elliptical orbital and rotational motions, as well as those from the Jovian and Lunar perturbations, are shown as functions of θ (the angle of the pulsar from the orbital plane normal) and ϕ (the angle of the pulsar from the x-axis in the orbital plane), for $t = 6$ lunar months (6×29.5 d), $\alpha = \pi/2$ and $f_0 = 1$ kHz (see Fig. 2). In each plot, the maximum phase shift occurs near $\theta = \pi/2$ (when the pulsar direction is in Earth's orbital plane). The ϕ-value at the maximum is more variable. The approximate maximum phase shifts for the above perturbations are, respectively, 6.3×10^6, 3.2×10^6, 1.8×10^2, 2.6×10^4, and 1.2×10^2 rad. For comparison, the phase of the signal ($2\pi f_0 t$) in this time period without Doppler shifts is 9.6×10^{10} rad.

We have also plotted the positional errors (in units of cycles of a 1 kHz gravitational wave) relative to the exact Keplerian orbit, both for the circular approximation and that of (8) (see Fig. 3). The estimation of the orbital phase is about four orders of magnitude more accurate using (8), a substantial reduction in the error. Figure 3 shows the positional errors resulting from ignoring the effects of the Moon and Jupiter (assuming the initial phases are such that the errors start at zero) for comparison. Both perturbers have substantial effects on the inertial position of the Earth and must be considered to properly compute the expected phase shift of the GW signal.

Fig. 2. Plot of phase shift due to (*a*) a circular orbit, (*b*)an elliptical orbit, (*c*) the Earth's rotation, (*d*) Jupiter, and (*e*) the Moon, as a function of θ and ϕ.



5. The Fourier transform of the perturbation-corrected gravitational-wave pulsar signal

In Valluri et al. [3], we arrive at an analytic expression for the Fourier transform of a GW signal, considering the perturbations above

$$\widetilde{h}(f) = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \sum_{N=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_0 \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6 \tag{13}$$

where

$$\psi_0(r, s, N, n, l, m, \alpha, \theta, \phi) = 4\pi i^{r+s+N+l} Y_{lm}(\theta, \phi) N_{lm} P_l^m(\cos\alpha)\, \mathrm{e}^{-i(r+s+N)\phi}$$

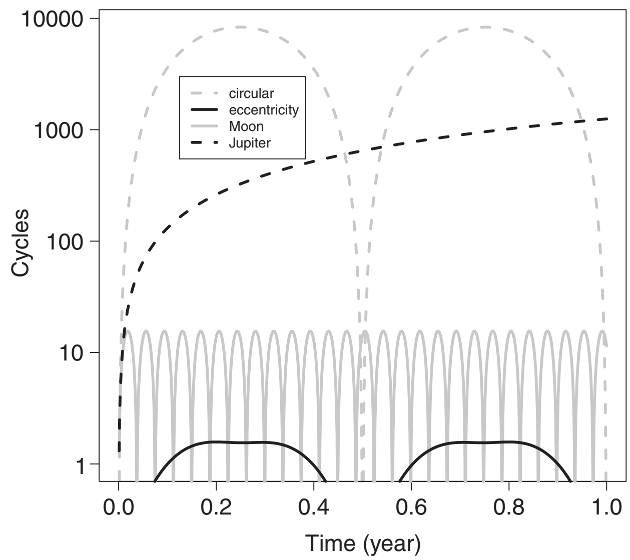
$$\psi_1(n, \theta, \phi, f_0) = T_{\mathrm{Er}} \sqrt{\frac{\pi}{2}} \exp\left[-i \frac{2\pi f_0 a}{c} \sin\theta \cos\phi - in\phi\right] i^n J_n\left(\frac{2\pi f_0}{c} a \sin\theta \left(1 - \frac{3}{8}\mathrm{e}^2\right)\right)$$

$$\psi_2(l, n, m, r, s, N, f_0, \omega) = \left\{\frac{1 - \mathrm{e}^{i\pi(l-B_T)R}}{1 - \mathrm{e}^{i\pi(l-B_T)}}\right\} 2\mathrm{e}^{-iB_T\frac{\pi}{2}} \frac{1}{2^{2l+1}}$$

$$\psi_3(k, l, m, n, r, s, N, f_0, \omega) = k^{l+(1/2)} \frac{\Gamma(l+1)}{\Gamma\left(l+\frac{3}{2}\right) \Gamma\left(\frac{l+B_T+2}{2}\right) \Gamma\left(\frac{l-B_T+2}{2}\right)}$$

$$\psi_4(r, s, N, \theta, f_0) = J_r\left(\frac{2\pi f_0}{2c} ae \sin\theta\right) J_s\left(\frac{2\pi f_0}{c} R_{\mathrm{J}} \sin\theta\right) J_N\left(\frac{2\pi f_0}{c} R_{\mathrm{EM}} \sin\theta\right)$$

Fig. 3. The difference between the Earth's position as computed by a circular approximation versus an exact Keplerian solution, measured in cycles of a 1 kHz gravitational wave over the course of 1 year (broken grey line). The (much reduced) difference when (8) is used is shown as a continuous black line. Contributions to the motion of the Earth due to the Moon and Jupiter, assumed to be on circular orbits that are coplanar with the Earth'sorbit, are shown for comparison.



$$\psi_5(\theta, \phi, f_0) = \exp\left[-\frac{i2\pi f_o}{c}\sin\theta\cos\phi\left(\frac{3}{2}ae + R_J + R_{\mathrm{EM}}\right)\right]$$

$$\psi_6(k, l, m, n, r, s, N, f_0, \omega) = {}_1F_3\left(l+1; l+\frac{3}{2}, \frac{l+B_T+2}{2}, \frac{l-B_T+2}{2}; \frac{-k^2}{16}\right)$$

Here c is the speed of light, and $\omega_0 = 2\pi f_0$. Also,

$$B_T = 2\left(\frac{\omega-\omega_0}{\omega_{\mathrm{rot}}} + \frac{m}{2} + \frac{n}{2}\frac{\omega_{\mathrm{orb}}}{\omega_{\mathrm{rot}}} + \frac{r}{2}\frac{2\omega_{\mathrm{orb}}}{\omega_{\mathrm{rot}}} + \frac{s}{2}\frac{\omega_{\mathrm{J}}}{\omega_{\mathrm{rot}}} + \frac{N}{2}\frac{\omega_{\mathrm{M}}}{\omega_{rot}}\right) \quad \text{and} \quad k = \frac{4\pi f_0 R_{\mathrm{E}}\sin(\alpha)}{c}$$

The above analytic expression for the Fourier transform helps to provide a convenient search template for GW forms, and can be used in matched-filtering analysis for processing actual GW data obtained from detectors. The intensity of the signal, $\left|\tilde{h}(f)\right|^2$, can be expressed in analytic form from (13). A detailed numerical analysis of this expression for the variety of parameters, which can take huge values ($> 50\,000$), present in the GW signal is a challenge for high-performance parallel computation [8]. The numerical analysis, which has already determined that the summations converge, will also determine the speed of convergence. Feynman has discussed the diffraction pattern due to the factor in the curly brackets in the expression for ψ_2 that gives the resultant amplitude due to R equal oscillators ($R = 365$ in our calculations) [9].

6. Conclusions

We have estimated, in this paper, the approximate phase corrections to a received GW pulsar signal due to rotational and elliptical orbital motion of the Earth, approximate spindown effects, and perturbations due to Jupiter and the Moon. The plane wave expansion in spherical harmonics improves the numerical accuracy and convergence of analytic Fourier transforms, enables the spindown corrections associated with the GW signal, and also the estimation of parameters and distributions relevant to GW data analysis. The plane wave expansion in spherical harmonics has important applications, for example, in the study of the multipole moments in cosmic microwave background anisotropies [10]. The proper blend of analytic and numerical integration for accurate GW data analysis remains an interesting question to be explored further.

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