Chapter 7.   The Principle of Least Action

7.1   Force Methods vs. Energy Methods

We have so far studied two distinct ways of analyzing physics problems: force methods, basically consisting of the application of Newton's Laws, and energy methods, consisting of the application of the principle of conservation of energy (the conservations of linear and angular momenta can also be considered as part of this). Both have their advantages and disadvantages. More precisely, energy methods often involve scalar quantities (e.g., work, kinetic energy, and potential energy) and are thus easier to handle than forces, which are vectorial in nature. However, forces tell us more. The simple example of a particle subjected to the earth's gravitation field will clearly illustrate this.

We know that when a particle moves from position y_0 to y in the earth's gravitational field, the conservation of energy tells us that

$$\Delta K + \Delta U_{\text{grav}} = 0$$

$$\frac{1}{2}m\left(v^2 - v_0^2\right) + mg\left(y - y_0\right) = 0,$$

(7.1)

which implies that the final speed of the particle, of mass m, is

$$|v| = \sqrt{v_0^2 + 2g\left(y_0 - y\right)}.$$

(7.2)

Although we know the final speed v of the particle, given its initial speed v_0 and the initial and final positions y_0 and y, we do not know how its position and velocity (and its speed) evolve with time.

On the other hand, if we apply Newton's Second Law, then we write

$$-mg\mathbf{e}_y = ma\mathbf{e}_y$$

$$= m\frac{dv}{dt}\mathbf{e}_y.$$

(7.3)

This equation is easily manipulated to yield

$$v = \int_{t_0}^{t} dv$$

$$= -\int_{t_0}^{t} g\,d\tau + v_0$$

$$= -g\left(t - t_0\right) + v_0,$$

(7.4)

where v_0 is the initial velocity (it appears in equation (7.4) as a *constant of integration*). If we conveniently choose $t_0 \equiv 0$, then it follows that (note that v can be negative)

$$v = v_0 - gt, \tag{7.5}$$

while we note that we chose the y-axis to increase upward (as could also be guessed from equation (7.3)). We can integrate once more to get

$$
\begin{aligned}
y &= \int_0^t v\, d\tau \\
&= \int_0^t (v_0 - g\tau)\, d\tau \\
&= y_0 + v_0 t - \frac{1}{2} g t^2.
\end{aligned}
\tag{7.6}
$$

Equations (7.5) and (7.6) allow us to track the position and velocity of the particle at any time after $t = 0$, something impossible to glean from the conservation of energy alone. We can further manipulate these two equations by taking the square of equation (7.5)

$$
\begin{aligned}
v^2 &= v_0^2 - 2 v_0 g t + g^2 t^2 \\
&= v_0^2 - 2g \left(v_0 t - \frac{1}{2} g t^2 \right) \\
&= v_0^2 - 2g (y - y_0),
\end{aligned}
\tag{7.7}
$$

where equation (7.6) was inserted in the last step. Equation (7.7) is the same as equation (7.2) obtained through the principle of conservation of energy. We conclude that, although more difficult to apply, the force method is nonetheless significantly more powerful.

One is left to wonder whether there exists a method that would combine the ease of application of the energy methods but yield the power of the force methods. This is the topic we study for the rest of this chapter.

7.2   The Principle of Least Action and Newton's Second Law

Minimizing principles have been known and used for a long time in physics to solve problems of all kind. For example, the propagation of a beam of light between two points can be determined by minimizing the time of travel (the Principle of Least Time), or it is often found that a system will settle to rest in a configuration that minimizes it energy (e.g., a ball in a parabolic well). It is therefore not surprising that physicists have sought an overarching minimization principle from which all of physics could be derived from (including, for example, Newton's Laws).

As it turns out, there exists such a principle: **The Principle of Least Action**, which can be stated as follows

*The motion of a system from time t_1 to t_2 is such that the line integral (called the **action**)*

$$I = \int_{t_1}^{t_2} L \, dt, \qquad (7.8)$$

where $L = K - U$ (with K and U the kinetic and potential energies, respectively) is minimized for the actual path of the motion.

The quantity L is called the **Lagrangian**, after the French-Italian scientist Joseph-Louis Lagrange (1736–1813) who is responsible for establishing the theory that rests on the minimization of the action. Although we will soon show how equation (7.8) can be minimized, mathematically, to derive the so-called **Lagrange equations**, it is more instructive at first to see how we can use the a priori knowledge of Newton's Second Law to "guess" how the Lagrangian is such a fundamental physical quantity.

Let us then return to our earlier problem of the particle subjected to the earth's gravity, keeping in mind that we wish to attack it using scalar quantities, such as the kinetic and gravitational potential energies. The question is as follows: Given that Newton's Second Law specifies that (see equation (7.3))

$$m\frac{d^2y}{dt^2} + mg = 0, \qquad (7.9)$$

how could we use the kinetic and gravitational potential energies K and U to arrive at the same result? The first part to the answer rests with the relation we established in Chapter 2 between the gravitational force and the potential energy, i.e.,

$$\mathbf{F} = -\nabla U, \qquad (7.10)$$

or in this case

$$
\begin{aligned}
F &= -\frac{dU}{dy}. \\
&= -\frac{d(mgy)}{dy} \\
&= -mg.
\end{aligned}
\qquad (7.11)
$$

Next we consider the inertial term $m\,d^2y/dt^2$ in equation (7.9) and rewrite it as

$$m\frac{d^2y}{dt^2} = m\frac{dv}{dt}. \qquad (7.12)$$

Since we note that equation (7.12) is a function of the speed v it is quite clear that we cannot, once again, use the potential energy for this term as it only depends on the position y. On the other hand, the kinetic energy can be advantageously considered because

$$K = \frac{1}{2}mv^2,$$ (7.13)

while

$$\frac{dK}{dv} = mv$$ (7.14)

and

$$\frac{d}{dt}\left(\frac{dK}{dv}\right) = m\frac{dv}{dt}$$
$$= m\frac{d^2y}{dt^2}.$$ (7.15)

It follows from equations (7.11) and (7.15) that we can write

$$\frac{d}{dt}\left(\frac{dK}{dv}\right) + \frac{dU}{dy} = m\frac{d^2y}{dt^2} + mg$$
$$= 0.$$ (7.16)

We thus recover equation (7.3), which is what we sought to accomplish.

We now introduce a small change in notation by putting a 'dot' above a variable to denote its time derivative. That is,

$$v = \frac{dy}{dt}$$
$$\equiv \dot{y}.$$ (7.17)

Finally, we also note that K is solely a function of the velocity $\dot{y}$ and independent of the position y, and that conversely, as was mentioned earlier, U is only a function of the position y and not $\dot{y}$, such that

$$\frac{dK}{dy} = \frac{dU}{d\dot{y}} = 0.$$ (7.18)

We therefore introduce a new function (i.e., the Lagrangian)

$$L(y,\dot{y}) \equiv K(\dot{y}) - U(y) \tag{7.19}$$

and write the equation of motion (7.16) as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0. \tag{7.20}$$

This is the general form of the equation of motion we were seeking.[1] We can now verify that this applies equally well to other physical situations in general (and not only a particle in a gravitational field) by testing it on well-known problems.

7.2.1 Exercises

1. **The Harmonic Oscillator.** A particle of mass m is attached to a massless spring of force constant k, and allowed to slide without friction on a horizontal plane. Find the mass' equation of motion.

Solution.

The kinetic energy of the system is

$$K = \frac{1}{2}m\dot{x}^2, \tag{7.21}$$

while the potential energy is that stored in the spring

$$U = \frac{1}{2}kx^2. \tag{7.22}$$

The Lagragian is therefore

$$\begin{aligned} L &= K - U \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2, \end{aligned} \tag{7.23}$$

while

---

[1] In equation (7.20) we used the *partial derivatives* $\partial L/\partial y$ and $\partial L/\partial \dot{y}$ instead of *total derivative* dL/dy and $dL/d\dot{y}$ to express the fact that the Lagrangian $L = L(y,\dot{y})$ is a function of both the position and the velocity. This is convention and notation used in mathematics to make this situation explicit.

$$\frac{\partial L}{\partial x} = -\frac{k}{2}\frac{\partial(x^2)}{\partial x}$$

$$= -kx \tag{7.24}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{m}{2}\frac{d}{dt}\left[\frac{\partial(\dot{x}^2)}{\partial \dot{x}}\right]$$

$$= \frac{m}{2}\frac{d(2\dot{x})}{dt} \tag{7.25}$$

$$= m\ddot{x}.$$

The equation of motion is then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} + kx \tag{7.26}$$

$$= 0,$$

as could easily be verified using Newton's Second Law. This equation of motion can be integrated to yield

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\cdot t + \phi\right), \tag{7.27}$$

with A and ϕ some constants that depend on the initial conditions (i.e., position and speed) when the mass was released and allowed to move.

2. **The Sliding Disk.** Find the equation of motion for a disk of mass m and radius R that is sliding down a frictionless ramp, inclined by an angle α, from rest at a height h.

Solution.

The kinetic energy of the disk is

$$K = \frac{1}{2}m\dot{y}^2, \tag{7.28}$$

with y its position along the incline ($y = 0$ at the top and increasing downward). The potential energy is

$$U = mg\left[h + R - y\sin(\alpha)\right], \tag{7.29}$$

and the Lagrangian

$$L = K - U$$
$$= \frac{1}{2}m\dot{y}^2 - mg\left[h + R - y\sin(\alpha)\right]. \tag{7.30}$$

We now calculate

$$\frac{\partial L}{\partial y} = -mg\left[\frac{\partial(h+R)}{\partial y} - \left(\frac{\partial y}{\partial y}\right)\sin(\alpha)\right]$$
$$= mg\sin(\alpha), \tag{7.31}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{m}{2}\frac{d}{dt}\left[\frac{\partial(\dot{y}^2)}{\partial \dot{y}}\right]$$
$$= \frac{m}{2}\frac{d(2\dot{y})}{dt} \tag{7.32}$$
$$= m\ddot{y}.$$

The equation of motion for the disk's centre of mass is therefore

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} - mg\sin(\alpha)$$
$$= 0, \tag{7.33}$$

as could easily be verified using Newton's Second Law. This equation of motion can be integrated to yield

$$y(t) = h + R + \frac{1}{2}g\sin(\alpha)t^2. \tag{7.34}$$

Note that we did not make use of any forces (e.g., the normal force that the incline applies to the disk) to solve the problem.

3. **The Rolling Disk.** Find the equation of motion for the position of the centre of mass of a disk of mass m and radius R that is rolling without slipping down a ramp, inclined by an angle α, from rest at a height h (see Figure 1).

Solution.

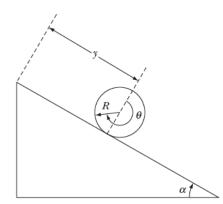The kinetic energy of the disk is the sum of translational and rotational components

Figure 1 - A disk rolling on an incline plane without slipping.

$$K = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2$$
$$= \frac{1}{2}m\dot{y}^2 + \frac{1}{4}mR^2\dot{\theta}^2,$$
(7.35)

where $I = mR^2/2$ is the moment of inertia of the disk about its axis of rotation and $\dot{y}$ is the velocity of its centre of mass. But we can also see from the figure that

$$y = R\theta$$
(7.36)

and therefore

$$\dot{y} = R\dot{\theta}.$$
(7.37)

Inserting equation (7.37) in equation (7.35) we get

$$K = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}m\dot{y}^2$$
$$= \frac{3}{4}m\dot{y}^2.$$
(7.38)

The potential energy is as calculated in equation (7.29) of Prob. 2

$$U = mg\left[h + R - y\sin(\alpha)\right],$$
(7.39)

and the Lagrangian

$$L = K - U$$

$$= \frac{3}{4} m \dot{y}^2 - mg \left[h + R - y \sin(\alpha) \right]. \qquad (7.40)$$

We now calculate

$$\frac{\partial L}{\partial y} = -mg \left[\frac{\partial (h+R)}{\partial y} - \left(\frac{\partial y}{\partial y} \right) \sin(\alpha) \right] \qquad (7.41)$$

$$= mg \sin(\alpha),$$

and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) = \frac{3}{4} m \frac{d}{dt} \left[\frac{\partial (\dot{y}^2)}{\partial \dot{y}} \right]$$

$$= \frac{3}{4} m \frac{d(2\dot{y})}{dt} \qquad (7.42)$$

$$= \frac{3}{2} m \ddot{y}.$$

The equation of motion for the crate is therefore

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = \frac{3}{2} m \ddot{y} - mg \sin(\alpha) \qquad (7.43)$$

$$= 0.$$

This equation of motion can be integrated to yield

$$y(t) = h + R + \frac{1}{3} g \sin(\alpha) t^2. \qquad (7.44)$$

We see that, when compared with equation (7.34) for the sliding disk of Prob. 2, the rotation of the disk brings an additional factor of $2/3$ in the position of the disk. That is, the disk goes slower down the incline by a factor of $2/3$.

4. **The Pendulum.** Find the equation of motion for the angular position θ of the pendulum made of a light string of length ℓ and a mass m shown in Figure 2.

Solution.

The kinetic energy of the system is

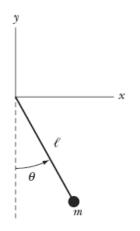**Figure 2 -** A simple pendulum of mass m and length ℓ.

$$
\begin{aligned}
K &= \frac{1}{2} I \dot{\theta}^2 \\
&= \frac{1}{2} m \ell^2 \dot{\theta}^2,
\end{aligned}
\tag{7.45}
$$

while the potential energy is given by

$$
U = -mg\ell \cos(\theta).
\tag{7.46}
$$

The Lagrangian is therefore

$$
\begin{aligned}
L &= K - U \\
&= \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell \cos(\theta),
\end{aligned}
\tag{7.47}
$$

with the needed derivatives

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= -mg\ell \sin(\theta) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) &= \frac{1}{2} m\ell^2 \frac{d}{dt}\left[\frac{\partial(\dot{\theta}^2)}{\partial \dot{\theta}} \right] \\
&= m\ell^2 \ddot{\theta}.
\end{aligned}
\tag{7.48}
$$

The equation of motion is thus

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= m\ell^2 \ddot{\theta} + mg\ell \sin(\theta) \\
&= 0.
\end{aligned}
\tag{7.49}
$$

For cases of small oscillations $\sin(\theta) \simeq \theta$, and the equation of motion becomes

$$\ddot{\theta} + \frac{g}{\ell}\theta \simeq 0, \qquad (7.50)$$

with for solution

$$\theta(t) \simeq A\cos\left(\sqrt{\frac{g}{\ell}} \cdot t + \phi\right). \qquad (7.51)$$

The quantities A and ϕ are some constants that depend on the initial conditions (i.e., position and velocity) when the mass was released and allowed to oscillate.

5. **The Coupled Oscillator.** The Lagrangian formalism is also very powerful to derive the equations of motion for systems that have more than one *degree of freedom*. That is, in the preceding problems we had to solve for only one variable (or degree of freedom; e.g., the position of the mass attached to a spring, the centre of mass position of a disk coming down an incline, or the angular displacement of a pendulum), but it is not uncommon that more than one variables are involved in the description of a physical system. For example, let us consider the coupled oscillator composed of three massless springs (two of force constant κ and one of constant κ_{12}) and two particles of similar mass M, as shown in Figure 3. We will use the Lagrangian formalism to simultaneously determine the equations of motion for the positions x_1 and x_2 of the two masses.

Solution.

The kinetic energy of the system is made of the sum of the corresponding energies of the individual particles

$$K = \frac{1}{2}M\left(\dot{x}_1^2 + \dot{x}_2^2\right), \qquad (7.52)$$

while the potential energy of the system must include a term, for the middle spring of force constant κ_{12}, that depends on the difference in the positions of the two particles

$$U = \frac{1}{2}\kappa\left(x_1^2 + x_2^2\right) + \frac{1}{2}\kappa_{12}\left(x_2 - x_1\right)^2. \qquad (7.53)$$

The Lagrangian is then

$$L = \frac{1}{2}M\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}\kappa\left(x_1^2 + x_2^2\right) - \frac{1}{2}\kappa_{12}\left(x_2 - x_1\right)^2, \qquad (7.54)$$

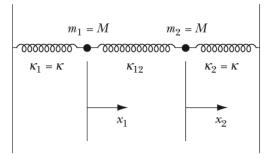with the needed derivatives

Figure 3 - Two masses connected to each other by a spring κ_{12} and to fixed points by other springs κ.

$$\frac{\partial L}{\partial x_1} = -\kappa x_1 + \kappa_{12}(x_2 - x_1)$$

$$\frac{\partial L}{\partial x_2} = -\kappa x_2 - \kappa_{12}(x_2 - x_1)$$

(7.55)

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = M\ddot{x}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = M\ddot{x}_2.$$

(7.56)

We thus have two equations of motion, one for x_1 and another for x_2,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = M\ddot{x}_1 + (\kappa + \kappa_{12})x_1 - \kappa_{12}x_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = M\ddot{x}_2 + (\kappa + \kappa_{12})x_2 - \kappa_{12}x_1 = 0.$$

(7.57)

Determining the solution for such a set of coupled differential equations is well beyond the scope of of our discussion, but it can be shown to be

$$x_1(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2)$$

$$x_2(t) = -A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2),$$

(7.58)

with

$$\omega_1 = \sqrt{\frac{\kappa + 2\kappa_{12}}{M}}, \quad \omega_2 = \sqrt{\frac{\kappa}{M}}, \tag{7.59}$$

and where A_1, A_2, ϕ_1, and ϕ_2 are constants that depend on the initial conditions.

7.2.2 Minimization of the Action[2]

Although we have successfully applied Lagrange's equation to several physical systems, we have yet to prove that it stems from an overarching principle involving the minimization of the action integral defined in equation (7.8). We thus end this chapter by accomplishing this task.

In general, for a function $f(x)$ to exhibit a minimum at some point x_0 it must be that its slope vanishes at that point.[3] That is, the variation δf in the function should verify that

$$\delta f = \frac{df}{dx} \delta x = 0 \tag{7.60}$$

about x_0. If the function depends on more than one variable, e.g., $f = f(x, y)$, then both derivatives df/dx and df/dy must vanish about a point (x_0, y_0) for a minimum to occur there. And we find that, upon using the so-called **chain rule** of calculus,

$$\delta f = \frac{df}{dx} \delta x + \frac{df}{dy} \delta y = 0 \tag{7.61}$$

about (x_0, y_0). We will apply this technique to find the minimum of the action to the simpler case of a system with only one degree of freedom x (a generalization to several degrees of freedom is straightforward), by first nothing that it is a function of both x and $\dot{x}$. That is,

$$
\begin{aligned}
\delta I(x, \dot{x}) &= \delta \int_{t_1}^{t_2} L(x, \dot{x}) dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right] dt \\
&= 0.
\end{aligned}
\tag{7.62}
$$

---

[2] This section contains advanced mathematical concepts on which you will not be tested; it is solely provided for completeness.

[3] A maximum or an inflection point in $f(x)$ will also have vanishing derivatives; to ensure that a minimum (maximum) occurs we should verify that the second derivative $d^2 f/dx^2$ is positive (negative). We will assume in our analysis that a minimum occurs, although this should, in principle, be verified.

In equation (7.62) we applied the chain rule, as in equation (7.61), but considering x and $\dot{x}$ as the independent variables (like x and y were in equation (7.61)). We now transform the quantity $\delta\dot{x}$ as follows

$$
\begin{aligned}
\delta\dot{x} &= \delta\left(\frac{dx}{dt}\right) \\
&= \delta\left[\lim_{\Delta t \to 0}\frac{x(t+\Delta t)-x(t)}{\Delta t}\right] \\
&= \lim_{\Delta t \to 0}\frac{\delta x(t+\Delta t)-\delta x(t)}{\Delta t} \\
&= \frac{d(\delta x)}{dt}.
\end{aligned}
\tag{7.63}
$$

The second term in the integrand of equation (7.62) can therefore be written as

$$
\int_{t_1}^{t_2}\frac{\partial L}{\partial\dot{x}}\delta\dot{x}\,dt = \int_{t_1}^{t_2}\frac{\partial L}{\partial\dot{x}}\cdot\frac{d(\delta x)}{dt}\,dt.
\tag{7.64}
$$

We need to further transform this integral. But before we do so, let us consider the following derivative

$$
\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{x}}\cdot\delta x\right] = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)\cdot\delta x + \frac{\partial L}{\partial\dot{x}}\cdot\frac{d(\delta x)}{dt},
\tag{7.65}
$$

since we know that the derivative of a product of two functions follows the rule

$$
\frac{d}{dt}(uv) = \left(\frac{du}{dt}\right)v + u\left(\frac{dv}{dt}\right).
\tag{7.66}
$$

Inserting equation (7.65) in equation (7.64) we have

$$
\begin{aligned}
\int_{t_1}^{t_2}\frac{\partial L}{\partial\dot{x}}\delta\dot{x}\,dt &= \int_{t_1}^{t_2}\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{x}}\cdot\delta x\right]dt - \int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)\cdot\delta x\right]dt \\
&= \int_{t_1}^{t_2}d\left[\frac{\partial L}{\partial\dot{x}}\cdot\delta x\right] - \int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)\cdot\delta x\right]dt \\
&= \left[\frac{\partial L}{\partial\dot{x}}\cdot\delta x\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)\cdot\delta x\right]dt.
\end{aligned}
\tag{7.67}
$$

We now note, however, that the Principle of Least Action considers variations from the true path of motion between *two fixed points* (x_1,t_1) and (x_2,t_2) at which the variations $\delta x = 0$. That is,
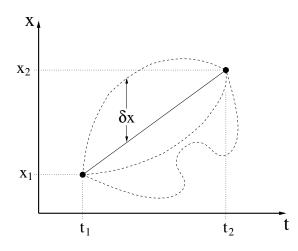
Figure 4 – Variations δx about the true path of motion (solid line) between to fixed points (x_1, t_1) and (x_2, t_2) at which $\delta x = 0$.

$$\delta x(t_1) = \delta x(t_2) \\ = 0.$$

(7.68)

This is shown in Figure 4. *The action is evaluated between these fixed points for all curves by integrating the corresponding Lagrangians; the true path of motion is the one for which the action is minimized.* Inserting equation (7.68) in the last of equations (7.67) we find

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \, dt = -\int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \cdot \delta x \right] dt,$$

(7.69)

and the variation of the action δI becomes (from equation (7.62))

$$\delta I(x, \dot{x}) = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \right] \delta x \, dt \\ = 0.$$

(7.70)

Since equation (7.70) is valid for any variation δx, it must be that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0.$$

(7.71)

We recognize the *Lagrange Equation* we derived earlier from Newton's Second Law (see equation (7.20)). This equation amazingly emerges from the Principle of Least Action. It is to be noted that, although all the examples we previously considered assumed that the kinetic and potential energies were respectively only functions of the velocity and the position, it is not necessarily the case in general (for example, the potential energy for a charged particle subjected to an electromagnetic field does depend on its velocity).