

## SECTION B: MODELS FOR NUCLEAR STRUCTURE

In this section we develop theoretical models to account for the stability and structure of nuclei, and in particular for the binding energy  $B$ . This will be done initially in terms of empirical or semi-classical methods and then the QM effects are introduced in stages.

### B.1 Liquid drop model

This is the simplest model for the binding energy, and it is mainly due to Weizsäcker (~ 1935). As the name suggests, it makes use of the analogies between nuclei and charges liquid drops:

- ◆ Spherical shape (radius  $R$ ) with definite boundary.
- ◆ Constant density as in an incompressible fluid (consistent with  $R = r_0 A^{1/3}$ )
- ◆ The drops are charged due to the protons (total charge  $Ze$  assume to be uniformly distributed).

The contributions to  $B$  (the negative of the binding energy) are:

#### (1) Volume energy

This is analogous to the (volume) cohesive energy in a liquid, so it gives a term for  $B$  that is proportional to  $R^3$  and hence to  $A$ :

So: we write contribution to  $B = +a_1 A$  ( $a_1 = +ve$  constant)

#### (2) Surface energy

This is analogous to the surface tension effect or surface energy in a liquid, so it is proportional to the surface area ( $\sim R^2$ ).

So: contribution to  $B = -a_2 A^{2/3}$  ( $a_2 = +ve$  constant)

#### (3) Coulomb energy

This arises due to the electrostatic potential energy of a volume distribution of charges (assuming a total charge  $Ze$  distributed uniformly through a sphere).

$$\text{Total electrostatic energy} \propto \frac{(\text{charge})^2}{R} \propto \frac{Z^2}{A^{1/3}}$$

This will tend to decrease the value of  $B$ .

So: contribution to  $B = -a_3 Z^2 / A^{1/3}$  ( $a_3 = +ve$  constant)

#### (4) 'Out-of-balance' term

If we assume that there is a tendency for maximum stability when  $N$  and  $Z$  are close in value (as observed experimentally) then:

When  $N \neq Z$  we need a term that reduces the value of  $B$ . This can be represented in  $B$  by an extra term of the phenomenological form:  $-a_4 (N - Z)^2 / A$  ( $a_4 = +ve$  constant)

This expression can be justified using QM (see later).

To summarize, we have (so far) got

$$M(A, Z) = (Nm_n + Zm_p) - B/c^2 \quad \text{with}$$

$$B = a_1 A - a_2 A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(A - 2Z)^2}{A}$$

This result for  $B$  is sometimes called the “semi-empirical mass formula” and it gives a fairly good fit to data for  $B$  using just 4 adjustable parameters. The approximate values of the constants are

$$a_1 = 15.6, a_2 = 16.8, a_3 = 0.72, \text{ and } a_4 = 23.3 \quad (\text{all in MeV})$$

Sometimes a fifth term is added to the semi-empirical mass formula (mainly to model the preferences for  $N$  and  $Z$  to be even, rather than odd) – see later.

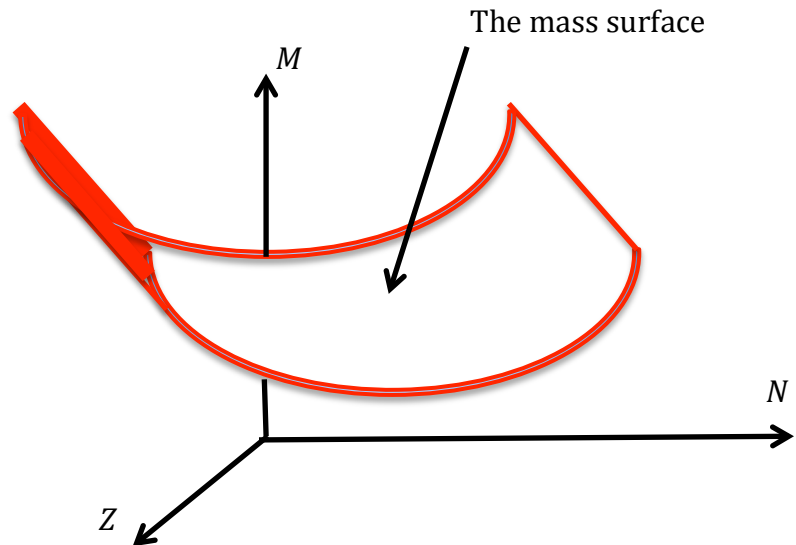
We now look at some practical applications of the liquid drop model:-

## The mass surface and beta decay

Suppose we make a 3D plot in which mass  $M$  is plotted vertically against  $N$  and  $Z$  plotted in the horizontal plane.

This produces a trough-like surface known as the “mass surface”.

If we now take any vertical cut (or section) chosen such that  $A = N + Z = \text{constant}$  Then the resulting curve (for  $M$  as a function of  $Z$  or  $N$ ) will be a curve with a minimum.

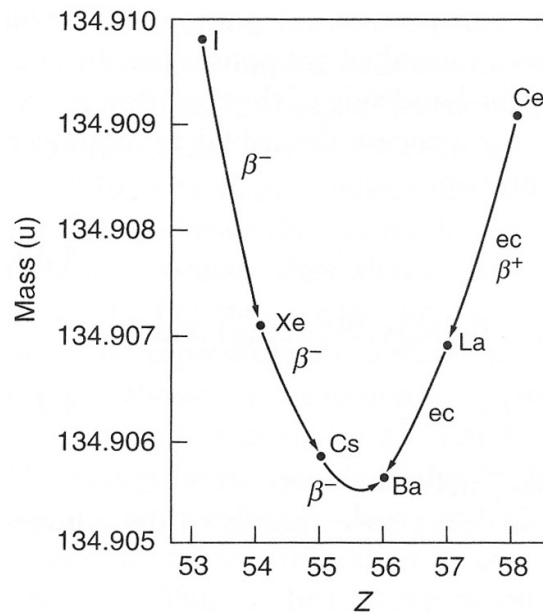


For example, in the case of  $A = 135$ , we have for mass  $M$  plotted versus  $Z$ :

Here  $M$  is expressed in atomic mass units:  
 $1 \text{ u} = (1/12) \text{ mass of neutral carbon } ^{12}\text{C}^6 \text{ atom.}$   
 The conversions of units are approximately:  
 $1 \text{ u} = 1.661 \times 10^{-27} \text{ kg} = 931.5 \text{ MeV}/c^2.$

The curve is called the “mass parabola”. Its minimum (or the integer  $Z$  closest to it) represents the most stable nucleus, which in this case is  $^{135}\text{Ba}^{56}$ .

Transitions to maximum stability can occur by the  $\beta$  decay processes mentioned earlier (which keep  $A$  constant). To this, we can add the related process of electron capture (ec).



$\beta^-$  decay:  $(A, Z) \rightarrow (A, Z+1) + e^- + \bar{\nu}$  ( $\bar{\nu}$  = antineutrino)

$\beta^+$  decay:  $(A, Z) \rightarrow (A, Z-1) + e^+ + \nu$  ( $\nu$  = neutrino)

electron capture:  $(A, Z) + e^- \rightarrow (A, Z-1) + \nu$

To the left of the minimum (low  $Z$ ) in the mass parabola, the  $\beta^-$  decay will be favoured, while to the right the  $\beta^+$  decay and ec processes will be favoured.

By comparing masses on each side of the above decay schemes (and ignoring any rest mass for the neutrino and antineutrino), we arrive at conditions for the decays to occur spontaneously:

For  $\beta^-$  decay,  $M(A, Z) > M(A, Z+1) + m_e$

For  $\beta^+$  decay,  $M(A, Z) > M(A, Z-1) + m_e$

For electron capture,  $M(A, Z) > M(A, Z-1) - m_e$

Obviously ec can occur when  $\beta^+$  decay is not possible energetically, but only if electrons are present. The electrons are usually from the K-shell (innermost electron shell of the atom)  $\rightarrow$  K-capture.

### Condition for a stable nucleus

For a given  $A$ , the stability condition is

$$\left(\frac{\partial M}{\partial Z}\right)_A = 0 \quad \text{for the bottom of the mass parabola}$$

Taking 
$$M = m_n(A - Z) + m_p Z - \frac{1}{c^2} \left[ a_1 A - a_2 A^{2/3} - \frac{a_3 Z^2}{A^{1/3}} - \frac{a_4 (A - 2Z)^2}{A} \right],$$

we get 
$$\left(\frac{\partial M}{\partial Z}\right)_A = m_p - m_n + \frac{2a_3 Z}{c^2 A^{1/3}} - \frac{4a_4 (A - 2Z)}{c^2 A}$$

Putting this equal to 0 (and taking  $m_p \approx m_n$ ) gives

$$Z = \frac{A}{2 + (a_3 / 2a_4) A^{2/3}}$$

If we had  $a_3/a_4 = 0$ , this would imply  $Z = \frac{1}{2}A$  (or  $Z = N$ ). The fact that  $a_3/a_4$  is nonzero and positive provides an explanation for the departure of the  $Z$  value from  $\frac{1}{2}A$  (as seen in the figure on page 5).

*Example:* estimate the  $Z$  for stability in the case of nuclei with  $A = 135$ , given the  $a_3$  and  $a_4$  values as before. We had  $a_3 = 0.72$ , and  $a_4 = 23.3$  (both in MeV), so  $a_3/2a_4$  is approximately 0.0155. The above stability formula then gives  $Z \approx 56.1$  (or 56 to the nearest integer), which is correct.

### Generalization of the liquid drop model to include isobaric stability

Recall that isobars are nuclei with the same  $A$  (such as  $^{50}\text{Ti}^{22}$  and  $^{50}\text{Cr}^{24}$ ).

First, consider the case of  $A$  even. There are 2 possibilities: *either*  $Z$  is even (implying  $N$  even) *or*  $Z$  is odd (implying  $N$  odd).

As mention in the previous section, observations show a tendency for even values of these integers. This is not so far taken into account in the liquid drop model, but an extra term can be added to model the behavior in a phenomenological way (such as to increase  $B$  if  $Z$  is even and reduce  $B$  if  $Z$  is odd):

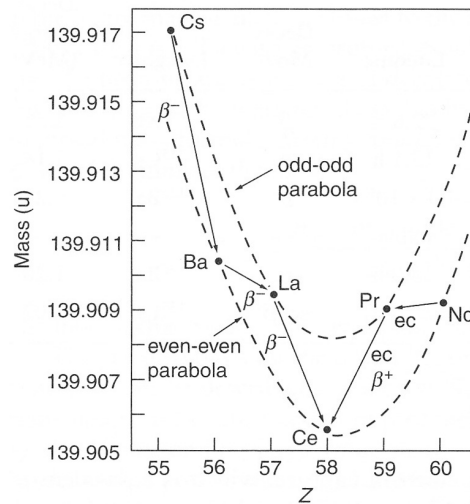
e.g., take 
$$B = a_1 A - a_2 A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(A - 2Z)^2}{A} + \begin{cases} +\Delta & \text{if } Z \text{ even, } N \text{ even} \\ -\Delta & \text{if } Z \text{ odd, } N \text{ odd} \end{cases}$$

where  $\Delta$  is an energy term ( $\Delta > 0$  and typically  $\sim 1$  MeV).

The stability conditions (against  $\beta$  decay and ec) can be analyzed by following a similar approach to before.

The difference is that there are now two mass parabolas, with curves separated vertically by an amount  $2\Delta/c^2$ .

This is illustrated here for the case of  $A = 140$ , where  $^{140}\text{Ce}^{58}$  is very stable compared with the nuclei for  $Z = 57$  and 59.



A similar treatment can be made to include the case of  $A$  odd (as well as  $A$  even as above):

$$B = a_1 A - a_2 A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(A-2Z)^2}{A} + \begin{cases} +\Delta & \text{if } Z \text{ even, } N \text{ even} \\ 0 & \text{if } Z \text{ even, } N \text{ odd, or vice-versa} \\ -\Delta & \text{if } Z \text{ odd, } N \text{ odd} \end{cases}$$

It is found from experiments that  $\Delta$  decreases as  $A$  increases (except for very small  $A$ ), so to a good approximation it is usual to take

$$\Delta = \frac{a_5}{A^{3/4}} \quad \text{with } a_5 = 34 \text{ MeV}.$$

As well  $\beta$  decay, the liquid drop model is also useful in analyzing other decay processes, like  $\alpha$  decay (where a nucleus emits an  $\alpha$  particle, which is just  ${}^4\text{He}^2$ ) or fission (where a nucleus splits into two fragments of roughly comparable size) — see later.

## B.2 Fermi gas model

This was one of the earliest attempts to include QM into the treatment of nuclei, mainly through the quantum statistics.

A stable nucleus is made up of neutrons and protons, which are both fermions (they obey Fermi-Dirac statistics and are subject to the Pauli Exclusion Principle — no two identical particles can occupy the same QM state).

This model treats the neutrons as one Fermi gas system and the protons as another separate system — but they occupy the same volume. All of the particles are non-interacting, so they have only KE. We can try to find the total energy using the Exclusion Principle applies, which tells us to fill up the lowest energy states one by one for each gas.

Consider first a single Fermi gas system with  $n_0$  particles of mass  $m$  in a volume  $V$ , and we work out its total energy. Given that a particle with momentum  $p$  will have energy  $p^2/2m$ , the total KE is

$$E = \int_0^{p_{\max}} \left( \frac{p^2}{2m} \right) D(p) dp$$

where  $p_{\max}$  is the momentum of the highest state, and  $D(p) dp$  tells us how many different states there are with momentum in the infinitesimal range between  $p$  and  $p + dp$ . It is called the *density-of-states function* and is given by

$$D(p) = \frac{Vp^2}{\pi^2 \hbar^3} \quad (\text{e.g., from QM solutions for states of a particle in a box})$$

*Proof:* Starting in the 1-dimensional case, the connection between particles and waves comes from de Broglie's hypothesis:  $p = \hbar k = 2\pi\hbar/\lambda$ . If the ends of the “box” (at  $x = 0$  and  $x = L$ ) are impenetrable we need to fit in an integer number of half waves (of wavelength  $\lambda$ ) into the length. This means

$$n(\lambda/2) = L \quad \text{with } n = 1, 2, 3, \dots$$

which leads to

$$p = n(\pi\hbar/L)$$

Therefore the separation of points in the 1-dimensional  $p$  space is  $(\pi\hbar/L)$ .

Generalizing to 3 dimensions, the volume per point in  $p$  space =  $(\pi\hbar/L)^3 = (\pi^3 \hbar^3 / V)$ .

The volume in  $p$  space between  $p$  and  $p + dp$

$$= \text{volume of thin shell (radius } p \text{ and thickness } dp)$$

$$= (1/8) \times 4\pi p^2 dp$$

So, the number of states is obtained by dividing by the volume factor for each state

$$= \frac{1}{2} \pi p^2 dp \div (\pi^3 \hbar^3 / V) = (V / 2\pi^2 \hbar^3) p^2 dp$$

Finally, a factor of 2 is included because of spin (for both neutrons and protons). This gives the result for  $D(p)$ .

Coming back to the total KE, we have

$$E = \frac{V}{2m\pi^2 \hbar^3} \int_0^{p_{\max}} p^4 dp = \frac{V}{10m\pi^2 \hbar^3} p_{\max}^5$$

We still need to find  $p_{\max}$ , which we can do as follows.

$$\text{Total number of particles} = n_0 = \int_0^{p_{\max}} D(p) dp = \frac{V}{\pi^2 \hbar^3} \int_0^{p_{\max}} p^2 dp = \frac{V}{3\pi^2 \hbar^3} p_{\max}^3$$

$$\text{Also, for a nucleus, } V = \frac{4}{3} \pi (r_0 A^{1/3})^3 = \frac{4}{3} \pi r_0^3 A$$

Finally, substituting for  $p_{\max}$  and  $V$  in the total KE gives

$$E = C \frac{n_0^{5/3}}{A^{2/3}} \quad \text{where} \quad C = \frac{3\hbar^2}{10mr_0^2} \left( \frac{9\pi}{4} \right)^{2/3}$$

Now in an actual nucleus of mass number  $A$  we have a gas of  $N$  neutrons and  $Z$  protons. Also, to a good approximation, we can take  $m_n = m_p$ , so constant  $C$  is the same. So the total energy for the nucleus in the Fermi gas model is

$$E = \frac{C}{A^{2/3}} (N^{5/3} + Z^{5/3})$$

To explore this prediction we consider that in most nuclei  $N$  and  $Z$  are roughly the same. Therefore we denote  $N - Z = \delta$ , where  $|\delta| \ll A$ . Since  $N + Z = A$ , we can replace  $N = \frac{1}{2}(A + \delta)$  and  $Z = \frac{1}{2}(A - \delta)$ , giving

$$\begin{aligned} E &= \frac{C}{2^{5/3} A^{2/3}} [(A + \delta)^{5/3} + (A - \delta)^{5/3}] = \frac{CA}{2^{5/3}} \left[ \left(1 + \frac{\delta}{A}\right)^{5/3} + \left(1 - \frac{\delta}{A}\right)^{5/3} \right] \\ &= \frac{CA}{2^{2/3}} \left[ 1 + \frac{5}{9} \left(\frac{\delta}{A}\right)^2 \right] \end{aligned}$$

Here we have used the Binomial expansion and kept just the leading-order terms. Rewriting we have

$$E = \frac{C}{2^{2/3}} \left[ A + \frac{5(N - Z)^2}{9A} \right]$$

Comparing with the liquid drop model, we recognize the first term above as being a contribution to the “volume” term in  $B$ , while the second term verifies the form of the assumed “out-of-balance” term.

In summary, the Fermi gas model has some limited success in using the particle statistics to estimate the kinetic energy of the neutrons and protons (giving a volume term and an out-of-balance term), but it ignores the quantization arising from confinement of the particles in the volume (radius  $R$ ) of the nucleus and interactions between the particles.

### B.3 Shell model

This is a QM method in which we use Schrödinger’s equation for each neutron or proton (individually) situated in a 3-dimensional potential well corresponding to the volume of the nucleus, i.e., we solve

$$\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + [E - V(\mathbf{r})] \psi(\mathbf{r}) = 0$$

where  $E$  is the energy of any nucleon (neutron or proton),  $m$  is its mass ( $= m_n$  or  $m_p$ ),  $\psi(\mathbf{r})$  is its wave function, and  $V(\mathbf{r})$  is its potential energy.

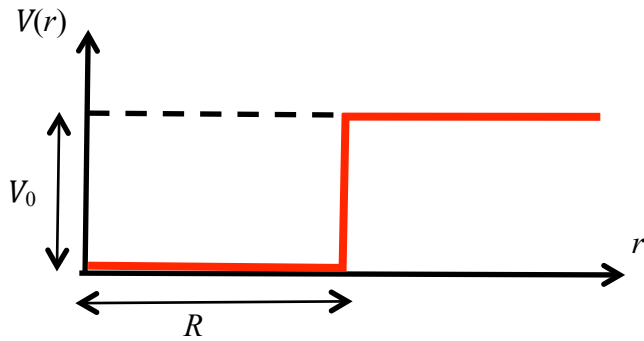
Use spherical polars:  $\psi(\mathbf{r}) = \psi(r, \theta, \phi)$

As a good first approximation we take the nucleus to be spherical in shape, with radius  $R$ , and  $V(r)$  to be spherically symmetric (depending on  $r$  only).

#### Form of nuclear potential well $V(r)$

We expect a difference for neutrons and protons, because of the charge in the latter case.

First, for neutrons, only the short-range nuclear forces are involved and the potential can be approximated as



We use  $R = r_0 A^{1/3}$  as before, but we also need an estimate for the magnitude of the well depth  $V_0$ .

This can be worked out using the Fermi gas model for the neutrons with their energy states filled from 0 to  $E_{\max}$ , where

$$E_{\max} = (p_{\max})^2 / 2m$$

and the previous estimate for  $p_{\max}$  was

$$p_{\max} = \frac{\hbar}{r_0} \left( \frac{9\pi n_0}{4A} \right)^{1/3} \quad \text{where here } n_0 \text{ is the number of neutrons.}$$

Taking  $n_0 = \frac{1}{2} A$  as a rough approximation, we have eventually 
$$E_{\max} = \frac{\hbar^2}{2mr_0^2} \left( \frac{9\pi}{8} \right)^{2/3}$$

This gives  $E_{\max} \sim 33 \text{ MeV}$  (independent of  $A$ ). Our estimate of the well depth is found by taking this value and then adding 8 MeV as the known average binding energy per nucleon.

So our final estimate is  $V_0 \sim 40 \text{ MeV}$

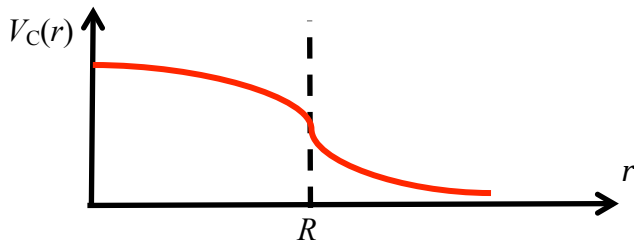
In practice, this is fairly close, e.g., for  $^{208}\text{Pb}^{82}$  we have approximately  $V_0 = 51 \text{ MeV}$  and  $R = 6.5 \text{ fm}$ .

Next, for the protons, the short-range nuclear force is essentially the same as in the neutron case (consistent with various scattering experiments), **BUT** the extra effect of Coulomb repulsion between any given proton and the other  $(Z-1)$  protons must be included. This will give an extra positive contribution  $V_C(r)$  to the potential.

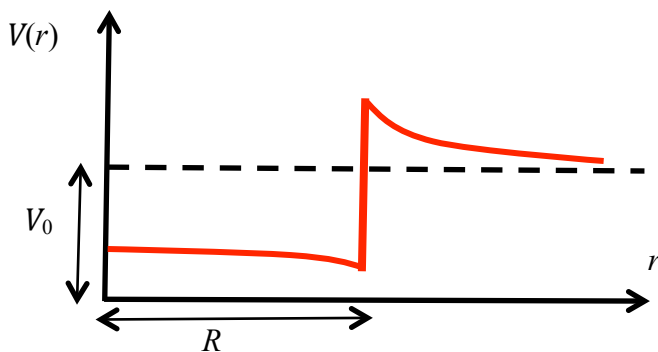
From basic electrostatics (using Gauss's law, etc) it can be shown that

$$V_C(r) = \frac{(Z-1)e^2}{4\pi\epsilon_0 r} \quad (\text{if } r \geq R) \quad \text{OR} \quad \frac{(Z-1)e^2}{4\pi\epsilon_0 R} \left( \frac{3R^2 - r^2}{2R^2} \right) \quad (\text{if } r < R)$$

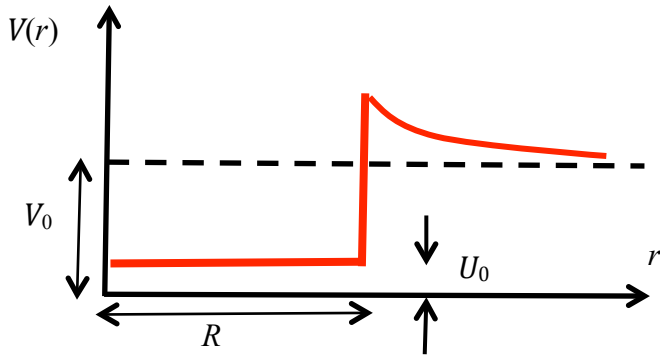
This looks like



This must be combined with the potential well for the nuclear forces to give:-



This is usually approximated (for convenience and also to account for some screening effects) within the potential well as:-



where  $U_0$  is the average value of the Coulomb term for  $r < R$ .  
For example, for  $^{208}\text{Pb}^{82}$  we have  $U_0 \sim 11$  MeV.

We now come back to Schrödinger's equation in spherical polar coordinates with a spherically symmetric  $V(r)$

$$\frac{\hbar^2}{2m} \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right\} \psi(r, \theta, \phi) + [E - V(r)] \psi(r, \theta, \phi) = 0$$

We look for separable solutions, i.e.,  $\psi(r, \theta, \phi) = u(r) T(\theta, \phi)$ . It can then be shown that the radial part and the angular part satisfy separate differential equations, which can each be solved.

The equation for the angular part is a standard equation with solutions known as *spherical harmonics* (denoted by  $Y$ ) that can be written in terms of *Legendre functions*:

$$T(\theta, \phi) = Y_{\ell m_\ell}(\theta, \phi) \sim P_\ell^{m_\ell}(\cos \theta) \exp(im_\ell \phi)$$

where  $\ell (= 1, 2, 3, \dots)$  and  $m_\ell (= -\ell, -\ell + 1, \dots, 0, \dots, \ell - 1, \ell)$  are the usual quantum numbers for angular momentum in QM. Some of the simplest Legendre functions are

$$P_0^0(\cos \theta) = 1, \quad P_1^0(\cos \theta) = \cos \theta, \quad P_1^{\pm 1}(\cos \theta) = \sin \theta, \quad P_2^0(\cos \theta) = (3 \cos^2 \theta - 1)/2$$

The radial function  $u(r)$  satisfies

$$\frac{\hbar^2}{2m} \frac{1}{r} \frac{d^2}{dr^2} (ru) + \left[ E - V(r) - \frac{\hbar^2 \ell(\ell + 1)}{2mr^2} \right] u = 0$$

If we know  $V(r)$  we can in principle solve for the radial function (although an analytic solution is possible only in a few special cases). The solution depends on  $\ell$  but not on  $m_\ell$ , so we denote the solutions by  $u_\ell(r)$ .

Degeneracy of states =  $2(2\ell + 1)$ , where the first factor of 2 is due to spin.

Hence there will be a *shell* structure, broadly analogous to the electron states in atoms. Before going further, it is useful to consider the experimental evidence for shell structure.

### Experimental evidence for shell structure in nuclei

It is found that, if  $N$  and/or  $Z$  have one of the following values:

$$2, \quad 8, \quad 20, \quad 28, \quad 50, \quad 82, \quad 126,$$

then the nucleus is particularly stable. The evidence for this is slightly better for  $N$  than for  $Z$ . The above numbers are often referred to as the *magic numbers* (number 14 is also sometimes included in the list).

(i) Estimates for relative abundance of nuclei in the solar system tend to have peaks corresponding to nuclei with  $N$  or  $Z$  magic.

(ii) If  $B/A$  for the binding energy per nucleon is plotted versus  $A$ , there are irregularities (small peaks) when  $N$  or  $Z$  are magic. [It is a small effect, typically an additional 1 – 2 MeV in the total  $B$ , compared with  $B/A \sim 8$  MeV].

(iii) There are peaks in the excitation energy associated with the first excited state of nuclei where  $N$  or  $Z$  is magic. [Excitation energy = energy difference between the ground state of a nucleus and one of its higher quantized states]. The peaks are  $\sim$  few MeV typically.

(iv) The stability of nuclei against decay (such as alpha and beta decay) is better when  $N$  or  $Z$  is magic.

We want to use the Shell model to *predict* the values for these magic numbers.

## Application of the Shell model for infinitely-deep potential wells

We will assume as before that the each nucleon moves independently of the other nucleons in a common potential well  $V(r)$  that is spherically symmetric, as discussed, and we use separation of variables as before. We know the angular part of the solution for the wave function, but so far we do not know the radial part and hence the energy.

We now take the simplification of an infinitely deep well (the limit of  $V_0 \rightarrow \infty$ ), so

$$V(r) = \begin{cases} 0 & \text{for } r \leq R \\ \infty & \text{for } r > R \end{cases} \quad \text{for both neutrons and protons.}$$

For  $r > R$  the wave function must vanish, so  $u_\ell(r) = 0$  in this region. For  $r < R$  we will have the radial equation

$$\frac{\hbar^2}{2m} \frac{1}{r} \frac{d^2}{dr^2}(r u_\ell) + \left[ E - \frac{\hbar^2 \ell(\ell+1)}{2mr^2} \right] u_\ell = 0$$

If  $\ell = 0$  ( $s$  state) this simplifies to

$$\frac{d^2}{dr^2}(r u_0) + \frac{2mE}{\hbar^2}(r u_0) = 0$$

The solutions for  $(r u_0)$  are of the form  $\sin(kr)$  and  $\cos(kr)$ , where we denote  $k^2 = 2mE / \hbar^2 > 0$ . However, the cos solution is impossible, because it would imply that  $u_0$  diverges for  $r = 0$ .

Therefore, we can write  $u_0(r) = A_0 \frac{\sin(kr)}{kr}$

where  $A_0$  is a normalization constant and  $k$  is related to the energy  $E$ . We now find the value(s) of  $k$  from the general boundary condition that  $u_\ell(r) = 0$  for  $r = R$ . This is so provided  $\sin(kR) = 0$ , giving energy levels that correspond to  $kR = \pi, 2\pi, 3\pi, 4\pi, \dots$  (for  $\ell = 0$ )

If  $\ell = 1$  ( $p$  state) we have  $\frac{d^2}{dr^2}(r u_1) + \left\{ k^2 - \frac{2}{r^2} \right\} (r u_1) = 0$  with  $k$  defined as before.

It can easily be verified that the correct solution is  $u_1(r) = A_1 \left\{ \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr} \right\}$

From the boundary condition at  $r = R$  this gives  $kR = 4.49, 7.73, \dots$  (found numerically)

Solutions for  $\ell = 2$  and larger can be found in a similar way. They all give a series of larger  $kR$  values, implying larger values for the energy  $E = (\hbar k)^2 / 2m$ . Recalling that the degeneracy of each energy level is  $2(2\ell + 1)$ , we can build up a table of energy levels and degeneracies - starting from the lowest levels which will be filled first. [Here  $n=1$  denotes the first solution in a sequence,  $n=2$  the second, and so on].

$kR$	$\ell$	$n$	No. in shell = $2(2\ell + 1)$	Total nucleons for complete shells	Nuclear state
3.14	0	1	2	2	$s$
4.49	1	1	6	8	$p$
5.76	2	1	10	18	$d$
6.28	0	2	2	20	$s$
6.99	3	1	14	34	$f$
7.73	1	2	6	40	$p$
8.18	4	1	18	58	$g$
.... and so on.					

The numbers of nucleons for complete shells are in the 5<sup>th</sup> column, and it is these that we want to compare with the magic numbers. It is seen that we have predicted some of the lower numbers (2, 8, 20), but the others are incorrect. Why?



### Improved versions of the Shell model

One limitation is the assumption that  $V(r) \rightarrow \infty$  for  $r > R$ . However, if the calculations are repeated for a finite well depth ( $\sim 40$  or  $50$  MeV), this makes only minor differences.

Also changing the shape of the well (e.g., by taking  $V(r) = Cr^2$  where  $C$  is a constant) does not lead to any real improvement in predicting the magic numbers.

It turns out that the main factor to include for a better theory is the effect of interactions between the spins of the neutrons or protons and their angular momentum. The interactions lead to a shift, and in some cases a splitting, of energy levels and hence a modification of the shell structure.

The interaction takes the form of an extra contribution (known as the *spin-orbit coupling*) to the potential of each nucleon:

$$V_{\text{so}} = -f \mathbf{L} \cdot \mathbf{S}$$

where  $\mathbf{L}$  is the orbital angular momentum of any nucleon,  $\mathbf{S}$  is its spin angular momentum, and constant  $f > 0$  is a strength factor. The total angular momentum  $\mathbf{J}$  of the nucleon is given by

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

where each angular momentum in QM is described by its quantum number ( $j$ ,  $\ell$ , and  $s$ , respectively), and

$$|\mathbf{J}|^2 = j(j+1)\hbar^2, \quad |\mathbf{L}|^2 = \ell(\ell+1)\hbar^2, \quad |\mathbf{S}|^2 = s(s+1)\hbar^2 = \frac{3}{4}\hbar^2$$

In the last part we used  $s = \frac{1}{2}$  for the spin of the neutron or proton. The usual QM rule for combining angular momenta gives

$$j = \begin{cases} \ell + \frac{1}{2} \text{ or } \ell - \frac{1}{2} & \text{if } \ell \geq 1 \\ \frac{1}{2} & \text{if } \ell = 0 \end{cases}$$

The effect on the energy levels can be found as follows:

$$\text{From } \mathbf{J} = \mathbf{L} + \mathbf{S} \text{ we get } \mathbf{J} \cdot \mathbf{J} = (\mathbf{L} + \mathbf{S}) \cdot (\mathbf{L} + \mathbf{S}) = \mathbf{L} \cdot \mathbf{L} + \mathbf{S} \cdot \mathbf{S} + 2\mathbf{L} \cdot \mathbf{S},$$

$$\text{so } V_{\text{so}} = -f \mathbf{L} \cdot \mathbf{S} = \frac{1}{2}f(|\mathbf{L}|^2 + |\mathbf{S}|^2 - |\mathbf{J}|^2) = \frac{1}{2}f\hbar^2\{\ell(\ell+1) + \frac{3}{4} - j(j+1)\}$$

This seen to be zero when  $\ell = 0$  (which implies  $j = \frac{1}{2}$ ), so there is no effect on a nuclear  $s$  state.

$$\text{For all other states we get } V_{\text{so}} = \begin{cases} -\frac{1}{2}f\hbar^2\ell & \text{if } j = \ell + \frac{1}{2} \\ \frac{1}{2}f\hbar^2(\ell+1) & \text{if } j = \ell - \frac{1}{2} \end{cases}$$

so there is a splitting of these energy levels by an amount  $f\hbar^2(\ell + \frac{1}{2})$ , which increases as  $\ell$  increases.

We can now modify the previous table for the nuclear states, keeping in mind that the spin-orbit term has no effect on a  $s$  state, gives only a very small splitting (usually neglected) for a  $p$  state, and gives a significant splitting for  $d, f, g$  states, etc.

$\ell$	$n$	$j$	No. of states = $(2j+1)$	Total nucleons for complete shells	Nuclear state
0	1	$1/2$	2	2	$s_{1/2}$
1	1	$3/2, 1/2$	$4 + 2 = 6$	8	$p_{3/2}, s_{3/2}$
2	1	$5/2$	6	14	$d_{5/2}$
2	1	$3/2$	4	18	$d_{3/2}$
0	2	$1/2$	2	20	$s_{1/2}$
3	1	$7/2$	8	28	$f_{7/2}$
3	1	$5/2$	6	34	$f_{5/2}$

... and so on.

We are now able to predict all the magic numbers at the appropriate places down the 5<sup>th</sup> column.

### Angular momentum and magnetic moment of nuclei

The Shell model can (in principle) be used to predict the angular momentum and magnetic moment of nuclei, which can then be compared with the experimental values.

For the angular momentum the following *pairing assumption* is usually made:

“An even number of nuclei of the same type in any given shell will always pair themselves such as to give zero overall angular momentum”.

In practice this assumption is good for nuclei in their ground state.

It then follows that the resultant angular momentum of the nucleus can come only from any odd (unpaired) particles, either  $n$  or  $p$ . With the Shell model (and the use of the last table) we can make predictions for the total angular momentum (in terms of its quantum number  $j$ ) for the ground state.

If we have  $N$  even and  $Z$  even  $\Rightarrow j = 0$

If  $N$  even and  $Z$  odd  $\Rightarrow j$  depends only of the odd  $Z$  value

If  $Z$  even and  $N$  odd  $\Rightarrow j$  depends only of the odd  $N$  value

In either case, if  $c$  denotes the odd number, then its value tells us which shell is only partially filled, so from the previous table we expect:

$$\begin{array}{ll} j = 1/2 & \text{for } c < 2 \\ j = 3/2 \text{ or } 1/2 & \text{for } 2 < c < 8 \\ j = 5/2 & \text{for } 8 < c < 14, \text{ etc} \end{array}$$

Finally, if  $N$  odd and  $Z$  odd (not many cases of stable nuclei with this property)  $\Rightarrow$  we can use both odd numbers to find a range of values for  $j$ . The process is:

Suppose  $j_n$  and  $j_p$  are the total angular momentum quantum numbers for the neutrons and protons separately (using the table), then the possible  $j$  values for the nucleus are

$$j_n + j_p, j_n + j_p - 1, \dots, |j_n - j_p|$$

Examples: a)  ${}^4\text{He}^2 \Rightarrow N = 2, Z = 2 \Rightarrow j = 0$

b)  ${}^{17}\text{O}^8 \Rightarrow N = 9, Z = 8$ , then  $N = 9 \Rightarrow j = 5/2$

c)  ${}^{10}\text{B}^5 \Rightarrow N = 5, Z = 5$ , then  $N = 5 \Rightarrow j_n = 1/2 \text{ or } 3/2$ , and  $Z = 5 \Rightarrow j_p = 1/2 \text{ or } 3/2$   
The possibilities for  $j$  are 3, 2, 1, or 0.

The predictions are correct for a) and b), and the actual value is 3 in c)

Overall, the predictions of the Shell model for the angular momentum are fairly good when compared with experiments. The few discrepancies are probably due to non-spherically symmetric terms in the potential  $V(\mathbf{r})$ .

As the magnetic moments of nuclei, the predictions of the Shell model are less impressive. They are OK in many cases, but there are now a lot of discrepancies. The failure is probably because the Shell model does not take account of the correlations between the motion of individual nuclei (i.e, the nucleons do not truly move independently of one another as usually assumed). These correlations also show up by leading to nuclei that are distorted in shape (not truly spherical).

## B.4 Collective model

This model attempts to include effects due the correlations in nucleon motion just mentioned and also to include nuclei that are non-spherical in shape.

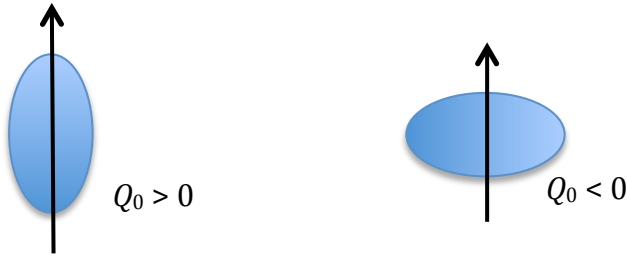
Typically, the shape of the nucleus is characterized by its electric quadrupole moment  $Q_0$ , which is usually defined by

$$Q_0 = \frac{1}{e} \int_{\text{nucleus}} \rho(\mathbf{r}) r^2 (3 \cos^2 \theta - 1) d^3 \mathbf{r}$$

where  $\rho$  is the volume charge density and  $\theta$  is the usual polar angle (taken relative to the axis for the total angular momentum).

If the charge density is spherically symmetric and the nuclear volume is a sphere, it can easily be checked (from the integration over angles) that  $Q_0 = 0$ .

For non-spherical nuclei the usual 2 cases are:



Most versions of the Collective model attempt to combine successful aspects of the Shell model and Liquid Drop model (which are based on opposite viewpoints):-

In the Liquid Drop model  $\Rightarrow$  The collective motion of the nucleons is described by a fluid analogy, while individual particle aspects (e.g., spin and orbital angular momentum) play no role. However, it can be generalized to non-spherical nuclei to describe quadrupole moments.

In the Shell model  $\Rightarrow$  QM solutions are found for individual nucleons moving independently. With the inclusion of spin-orbit interactions, it leads to a correct description of the angular momentum states and the occurrence of the magic numbers.

The basic form of the Collective model incorporates features from both of the above. It typically assumes:

1. The nucleus consists of an inner core of nucleons in filled shells as given by the Shell model, plus there are the remaining outer nucleons (“valence” nucleons) that behave like surface states in the Liquid Drop model.
2. The valence nucleons have a surface motion ( $\Rightarrow$  rotations effects, which are quantized just like rotations of molecules in QM). These effects lead to a distortion of the inner core and the entire nucleus.

The outcome is similar to having the Shell model, but with a modified  $V(\mathbf{r})$  that is no longer spherically symmetric and a volume that is now ellipsoidal rather than spherical.

For example, a sphere of radius  $R$  is described by the surface  $x^2 + y^2 + z^2 = R^2$

An ellipsoidal shape with the same volume is described by  $\alpha x^2 + \beta y^2 + (1/\alpha\beta)z^2 = R^2$

where either  $\alpha \neq 1$  or  $\beta \neq 1$  (or both) describe the amount of distortion. The simplest choice of potential is

$$V(r) = \begin{cases} 0 & \text{inside the ellipsoid} \\ \infty & \text{outside the ellipsoid} \end{cases}$$

The previous solutions of Schrödinger’s equation could be generalized, including the spin-orbit interaction. However, we would no longer find that the angular part of the solution is proportional to the spherical harmonics and Legendre functions (so the scheme of quantum numbers is different. The generalized Schrödinger’s equation also has to have terms added for rotational states.