SECTION 4 Electric Fields in Matter

This section (based on Chapter 4 of Griffiths) continues with the study of electrostatics but now in the presence of another material (such as an insulator or dielectric). The topics are:

Polarization

- Polarization
- The field of a polarized object
- Electric displacement
- Linear dielectrics

Conductors and Insulators

Most materials falls into two broad categories as regards their electrical properties:

• Conductors

These have free charges which move in response to external electric fields. When a conductor is placed in an electric field, the charges move to cancel out the applied external field, making the net field equal to 0 inside the conductor.



Insulators

These have charges which are **bound** to particular atoms or molecules.

When an insulator is placed in an electric field, the atoms and molecules become dipoles and line up with the field, partially cancelling the field in the insulator.



Unless the electric field is very strong, the electron cloud of a non-metallic atom is only moved a little in the direction opposite to the electric field. The atom then becomes a dipole with moment \mathbf{p} pointing in the direction of the electric field. If the field isn't too strong, we have a proportionality:

 $\mathbf{p} = \alpha \mathbf{E}$

where α is a constant called the polarizability. The polarizability is determined experimentally and it depends on the properties of the atom.







Nucleus and electron cloud displaced a small distance d

Molecules are more complicated, since they are typically asymmetric and can be categorized as:

Non-polar molecules (p = 0 when E = 0)





Polar molecules ($p \neq 0$ when E = 0)

For molecules, the polarizability is a function of direction of the electric field. In the case of carbon dioxide, the polarizability is different along the molecule's axis and perpendicular to it:

$$\mathbf{p} = \alpha_{\perp} \mathbf{E}_{\perp} + \alpha_{\parallel} \mathbf{E}_{\parallel}$$

In general, for asymmetric molecules, the polarizability depends on the field component in each direction: $p = \alpha E + \alpha E$

$$p_{x} = \alpha_{xx}E_{x} + \alpha_{xy}E_{y} + \alpha_{xz}E_{z}$$

$$p_{y} = \alpha_{yx}E_{x} + \alpha_{yy}E_{y} + \alpha_{yz}E_{z}$$

$$p_{z} = \alpha_{zx}E_{x} + \alpha_{zy}E_{y} + \alpha_{zz}E_{z}$$

$$QR$$

$$p = \alpha E$$

where **p** and **E** are column vector and α is a 3 × 3 matrix. Hence there is a polarizability matrix (or tensor) in these cases, instead of a scalar constant.

Polar molecules

Polar molecules, like water, which have built-in dipole moments, experience a torque when they are subjected to an electric field.



Consider a simple dipole of two charges q and -q separated by distance d. If the field **E** is uniform, the two forces are equal and opposite (so they cancel out), but they tend to align the molecule with the field.

$$\mathbf{F}_{+} = q\mathbf{E}$$
 $\mathbf{F}_{-} = -q\mathbf{E}$

The torque is:

 $\mathbf{N} = (\mathbf{r}_{+} \times \mathbf{F}_{+}) + (\mathbf{r}_{-} \times \mathbf{F}_{-}) = q(\mathbf{r}_{+} - \mathbf{r}_{-}) \times \mathbf{E} = q\mathbf{d} \times \mathbf{E}$ So a dipole with dipole moment **p** experiences a torque

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

For discussion in class: Show that the potential energy of the above dipole in the field E is given by -p.E

Force on an electric dipole

If the electric field is non-uniform, so that it varies from one side of the dipole to the other (which is a rather small effect typically for molecules), then there is a net force on the dipole:

$$\mathbf{F} = \mathbf{F}_{+} - \mathbf{F}_{-} = q(\mathbf{E}_{+} - \mathbf{E}_{-}) = q\Delta\mathbf{E}$$

If the dipole is very short (meaning vector \mathbf{d} is small), we can approximate the small change in any component of \mathbf{E} (say the *x* component) by:

$$\Delta E_x = \mathbf{d} \cdot \nabla E_x$$

Doing the same for the *y* and *z* components gives:

$$\Delta \mathbf{E} = \mathbf{d} \cdot \nabla \mathbf{E}$$

and therefore (since $\mathbf{p} = q\mathbf{d}$ for the dipole moment): $\mathbf{F} = \mathbf{p} \cdot \nabla \mathbf{E}$

Polarization of matter

In a material with neutral atoms or nonpolar molecules, when an electric field is applied, a tiny dipole is induced in each, which on average aligns with the electric field. In a material made of polar molecules, each dipole tends to align with the electric field.

In either case, the material becomes polarized (i.e., acquires a dipole moment throughout its volume), which can be expressed as:

Polarization \mathbf{P} = dipole moment per unit volume

Next, we look at the electric field or potential produced by the polarized matter.

The field of a polarized object

We have a dipole moment per unit volume in a polarized material. We already know from Section 3 that an individual dipole gives rise to an electric field. So the polarization of the material gives rise to an electric field.

To find this electric field, it's easier to use results for the potential. We start from the potential due to a single dipole, which is:



$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}_e \cdot \mathbf{p}}{r_e^2}$$

The potential outside a volume of polarized material is found by integrating over all the dipoles:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{\hat{\mathbf{r}}_c \cdot \mathbf{P}(\mathbf{r}')}{r_c^2} d\tau'$$

In principle, we can evaluate this integral directly for any polarized material; but there's actually a better way to look at the same calculation.

First, we know that:

$$\nabla'\left(\frac{1}{r_e}\right) = \frac{\hat{\mathbf{r}}_e}{r_e^2}$$

Here the prime on the gradient indicates that we're taking the derivative with respect to the source coordinates $(\mathbf{r}_e = \mathbf{r} - \mathbf{r}')$. So now we have:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \mathbf{P} \cdot \nabla' \left(\frac{1}{r_e}\right) d\tau'$$

We can next use integration by parts:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[\int_{\mathcal{V}} \nabla' \cdot \left(\frac{\mathbf{P}}{r_e} \right) d\tau' - \int_{\mathcal{V}} \frac{1}{r_e} (\nabla' \cdot \mathbf{P}) d\tau' \right]$$

Then, applying the divergence theorem to the first term in the bracket gives:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[\int_{\mathcal{S}} \frac{1}{r_e} \mathbf{P} \cdot d\mathbf{a}' - \int_{\mathcal{V}} \frac{1}{r_e} (\nabla' \cdot \mathbf{P}) d\tau' \right]$$

We can now re-interpret this result. Notice that the first term looks like the potential of a surface charge, where the charge per unit area is:

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

and the second term looks like the potential of a volume charge, where the charge density is:

$$o_b = -\nabla \cdot \mathbf{P}$$

So we can rewrite the potential in terms of these <u>bound charges</u> that are associated with the polarized material:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[\int_{\mathcal{S}} \frac{\sigma_b}{r_e} da' + \int_{\mathcal{V}} \frac{\rho_b}{r_e} d\tau' \right]$$

The bound charges are more than just a mathematical analogy: they correspond to real accumulations of charge in parts of the polarized material.

Look at two examples:



First, consider a cylinder with uniform polarization along the axis. Since the divergence of **P** will be zero inside, there will be no bound volume charge.

However, at the ends, the dipoles are lined up perpendicular to the surface:



There is a negative net surface charge on the left face of the cylinder, and a positive net surface charge on the right face. The total charge is not changed, only redistributed (if the material was neutral, the total charge is still zero).

Next, consider a sphere whose polarization is radially outwards (along the r direction):



Taking into account which way the dipoles point, it follows that here will be an excess of negative charge inside the sphere, where the polarization is large (theoretically it diverges near the centre), while the surface will have an excess of positive charge.

Again, if the material is neutral, the positive and negative charges will cancel overall.

Electric displacement

Total electric field around dielectrics

In the general case where a material is present, we want to calculate the total potential and the electric field that include the bound charges on a dielectric material as well as any other charges in the region (we will call them free charges to distinguish them from the induced or bound charge).

We can rewrite Gauss's law in terms of the total (volume) charge as

$$\rho = \rho_b + \rho_f \qquad \qquad \varepsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

Combining the terms with divergence, we have:

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

 $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

We can define a new vector **D**, called the electric displacement, which is the expression in parentheses:

Gauss's law then becomes:

$$\nabla \cdot \mathbf{D} = \rho_f$$
 or $\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f-\text{enc}}$

where Q_{f-enc} is the total free charge in our chosen volume.

This is Gauss's Law in the presence of dielectrics. We can use it to find the electric displacement in exactly the same way as we find electric field with the standard Gauss's law.

Problem 4.15 (from Griffiths) - for discussion in class

A thick spherical shell (inner radius *a*, outer radius *b*) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}}$$

where k is a constant and r is the distance from the centre. (There is *no free charge* in the problem). Find the electric field **E** in all 3 regions by two different methods:

(a) Locate all bound charge and use Gauss's law (as in chapter 2) to calculate the field.

(b) Use the modified Gauss's law (this chapter) to find **D**, and then deduce **E**. (The second method is *much quicker*).

Note of caution!

From Gauss's law it looks like the electric displacement **D** and the electric field **E** are analogous. However, we need to be careful because there is no analogue of Coulomb's law for **D**. This is because while the divergence of **D** looks similar to the divergence of **E**, the curls are not the same!

The curl of **E** always vanishes (and this property allowed us to define a scalar potential), but the curl of **D** can sometimes be nonzero. In fact:

$$\nabla \times \mathbf{D} = \mathcal{E}_0(\nabla \times \mathbf{E}) + (\nabla \times \mathbf{P}) = \nabla \times \mathbf{P}$$

The outcome is that you cannot always just use Gauss's law to calculate D (because it gives you the effect coming from the free charge, but doesn't tell you about the curl of P).

The rule-of-thumb is that, if the problem has spherical, cylindrical or plane symmetry, the curl of **D** vanishes and you can use the usual Gauss's law methods to solve it: in other situations it is more complicated.

Boundary conditions

Previously we looked at how the electric field changes at a surface where there is a charge per unit area:

$$E_{above}^{\perp} - E_{below}^{\perp} = \frac{1}{\varepsilon_0} \sigma$$
$$E_{above}^{\parallel} - E_{below}^{\parallel} = 0$$

Recall that the first result came from using Gauss's law and the second came from the vanishing of curl of E. It follows that the generalization of the boundary conditions in the case of D is

$$\begin{split} D_{above}^{\perp} - D_{below}^{\perp} &= \sigma_f \\ D_{above}^{\parallel} - D_{below}^{\parallel} &= P_{above}^{\parallel} - P_{below}^{\parallel} \end{split}$$

Linear dielectrics

For many materials, if the field is not too strong, the polarization is proportional to the electric field; these are called linear dielectrics. If P and E are in the same direction (parallel vectors), we can write

$$\mathbf{P} = \varepsilon_0 \boldsymbol{\chi}_e \mathbf{E}$$

where the proportionality factor χ_e is called the electric susceptibility of the medium; it is dimensionless. Here **E** is the total electric field; it includes parts coming from the free charge and from the bound charge induced in the dielectric.

We can now find some relationships involving **D**:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e) \mathbf{E}$$

This is often written as

 $\mathbf{D} = \varepsilon \mathbf{E}$

where ε is the permittivity of the material.

The relationship between parameters is

$$\varepsilon = \varepsilon_0 (1 + \chi_e)$$

Sometimes it is useful to write

 $\mathcal{E} = \mathcal{E}_r \mathcal{E}_0$

 $\varepsilon_r = 1 + \chi_e$

so that ε_r is dimensionless.

Then we have

and ε_r is called the relative permittivity (or the dielectric constant).

Some common	linear	dielectrics,
compared to vo	асиит.	•

	χe	<i>E</i> (C ² /N.m ²)	\mathcal{E}_r
Vacuum	0	8.85x10 ⁻¹²	1
Air	5.4x10 ⁻⁴	8.85x10 ⁻¹²	1.00054
Salt	4.9	5.22x10 ⁻¹¹	5.9
Ice	98	8.67x10 ⁻¹⁰	99

Fields in linear dielectrics

Inside a linear dielectric, the curl of P (and also of D) must vanish, since P is proportional to E. This argument does not hold at the boundary of a linear dielectric, and the curl of P may be nonzero there.

Nevertheless, there are cases where we can take advantage of the lack of curl of \mathbf{P} inside the linear dielectric: if the dielectric is much larger than the region we're interested in, or completely fills the space we're considering, we can use all the simplifications (because the surface becomes unimportant).

In the case that our space is entirely filled with a homogenous linear dielectric, we have:

$\nabla \cdot \mathbf{D} = \rho_f$ and $\nabla \times \mathbf{D} = 0$

These are exactly the equations for electric field in a vacuum, except for a factor of ε_0 . It means that **D** can be found from the free charges, just as though the dielectric were not there. So, if we define **E**_{vac} as the electric field that the same free charges would produce in the absence of any dielectric, then:

$$\mathbf{D} = \varepsilon_0 \mathbf{E}_{\text{vac}} \qquad \text{and} \\ \mathbf{E} = \frac{1}{\varepsilon} \mathbf{D} = \frac{1}{\varepsilon_r} \mathbf{E}_{\text{vac}}$$

For example, a point charge embedded in a large dielectric has a field:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \frac{q}{r^2} \hat{\mathbf{r}}$$

This modification to Coulomb's law means that the field is reduced compared to the vacuum case; the point charge is now "shielded" by induced charges in the polarized dielectric.

Most gases, liquids and amorphous solids (e.g., glasses) are approximately isotropic in their properties; so **P** is in the same direction as **E** and they have a single scalar constant susceptibility.

Crystalline solids, on the other hand, may polarize more easily in one direction (usually a crystal axis of symmetry) than another. In that case, the polarization is given by:

$$\mathbf{P} = \varepsilon_0 \boldsymbol{\chi}_e \mathbf{E}$$

where χ_e is now a susceptibility tensor (or 3×3 matrix) for the crystal, by analogy with the polarizability tensor for an asymmetric molecule, discussed earlier.

Boundary conditions with linear dielectrics

The bound charge density inside a linear dielectric is easily found:

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\varepsilon_0 \frac{\chi_e}{\varepsilon} \mathbf{D}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right) \rho_f$$

i.e., it is just proportional to the free charge.

In particular, the bound (or induced) charge density vanishes if the local free charge density is zero (no embedded charges in the dielectric). Then the only charge is on the surface of the dielectric (sort of like in the case of a conductor).

We could go ahead by using Laplace's equation to solve for the potential inside the dielectric - but first we would need the new boundary conditions to be able to solve Laplace's equation.

For the electric field, assuming there are now free charges σ_f at the surface, we have:

$$\varepsilon_{\text{above}} E_{\text{above}}^{\perp} - \varepsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$$
 (Note the permeability factors).

In terms of the potential we have

$$\varepsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \varepsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = \sigma_f$$

And, as before, the potential must be continuous across the boundary:

$$V_{\text{above}} = V_{\text{below}}$$

Energy in linear dielectrics

First we need to know how the capacitance changes when the capacitor is filled with a linear dielectric instead of vacuum. It is easy to see that the new capacitance is

$$C = \varepsilon_r C_{\rm vac}$$

(We use the definition C = Q/V and the fact that **E**, and hence *V*, are scaled by a factor of $1/\varepsilon_r$).

Since the energy of a capacitor (for any given voltage *V*) is $W = \frac{1}{2}CV^2$ it follows that *W* also is also increased by a factor of ε_r

In terms of the electric field the energy becomes

$$W = \frac{\varepsilon_0}{2} \int \varepsilon_r E^2 d\tau = \frac{\varepsilon_0}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau$$

Forces on dielectrics

We now show that dielectrics are attracted into regions of high electric fields; <u>but</u> it can be complicated to calculate the force on a dielectric.



We can take a shortcut by considering a simple example, i.e., slab of dielectric partially inside a parallel plate capacitor.

(Note: by considering the slab part way out, we are avoiding any difficulties with the fringing fields near the edges).

To find the force F we consider the work done dW when x is increased infinitesimally to x + dx. We will assume that the total charge on each plate stays constant at Q and -Q.

For the energy stored in the capacitor, we can write

$$W = \frac{1}{2} \frac{Q^2}{C}$$

So, for the infinitesimal change:

$$-Fdx = dW = d\left(\frac{1}{2}\frac{Q^2}{C}\right) = -\frac{Q^2}{2C^2}dC$$

In terms of the voltage V = Q/C, this becomes:

$$F = \frac{V^2}{2} \frac{dC}{dx}$$

We need an expression for *C*. The basic result for a simple parallel plate capacitor is that the capacitance is $\varepsilon A/d$, where *A* is the area of the plates.

In our example, area A varies as a function of x : we have one capacitor with area wx filled with vacuum and another with area w(l-x) filled with dielectric, where w is the other dimension of the plate, so

$$C = \frac{\varepsilon_0 w x}{d} + \frac{\varepsilon_r w (l - x)}{d}$$

This gives

$$\frac{dC}{dx} = \frac{\varepsilon_0 w}{d} - \frac{\varepsilon_0 \varepsilon_r w}{d} = \frac{\varepsilon_0 (1 - \varepsilon_r) w}{d} = \frac{-\varepsilon_0 \chi_e w}{d}$$

So the force is related to the susceptibility of the dielectric by

$$F = -\frac{\varepsilon_0 \chi_e w}{2d} V^2$$