SECTION 7
Electrodynamics

This section (based on Chapter 7 of Griffiths) covers effects where there is a time dependence of the electric and magnetic fields, leading to Maxwell's equations. The topics are:

- Electromotive force
- Electromagnetic induction
- Maxwell's Equations

Electromotive force

Changing electric and magnetic fields

Up to now, we have looked at the electric fields produced by stationary charges, and the magnetic fields produced by steady currents, which do not change as a function of time. There are many new physical effects in case where the fields vary rapidly with time; the inclusion of these phenomena will lead to Maxwell's equations of electromagnetism.

We first need to look in more detail at the properties of real conductors, starting with Ohm's law and the physical origins of conductivity

Ohm's law

We need to study how charge moves through matter. In most materials, the current is found to be proportional to the force **f** per unit charge:

$$\mathbf{J} = \sigma \mathbf{f}$$

Here σ is just a proportionality constant, known as the conductivity of the material, which is determined experimentally.

Often, tables of numerical values are expressed in terms of its reciprocal, the resistivity of the material:

$$\rho = 1/\sigma$$

Note: be careful not confuse the conductivity σ with the surface charge σ, or resistivity ρ with volume charge ρ.

Material	Resistivity (Ω•m)
Silver	$1.59\text{x}10^{-8}$
Copper	$1.68\text{x}10^{-8}$
Iron	$9.61\text{x}10^{-8}$
Graphite	$1.4\text{x}10^{-5}$
Salt water	$4.4\text{x}10^{-2}$
Silicon	$2.5\text{x}10^{3}$
Water	$2.5\text{x}10^{5}$
Glass	$10^{10} – 10^{14}$

The force driving the charges is related here to the total electromagnetic force:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

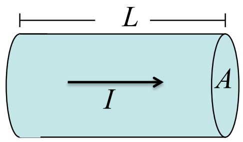
If the speed of the charges is small enough (and normally in matter this is the case), the magnetic force will be tiny compared to the electric force, so

$$\mathbf{J} = \sigma \mathbf{E}$$

This is Ohm's law, which is true in the special case of materials where the conducting charge responds linearly to the force and the charges are not too fast.

Note: we used the result in Section 2 that the electric field in a ideal conductor is zero (which would imply that σ is infinite). In a very good conductor (i.e., metals such as Cu and Ag), the conductivity is relatively large, so the electric field needed to move charges is very small.

For a resistive material, we can obtain the usual expression for the current flow in terms of the potential difference across the resistor:



If a resistor has length L and cross sectional area A (which is constant over the length), and there is a potential difference V, the current is:

$$I \;=\; JA \;=\; \sigma EA \;=\; \frac{\sigma A}{L}V$$

The resistance R depends on
the shape of the resistor. The general relationship is $V = IR$, so in this case we have

$$R = \frac{L}{\sigma A}$$

Notes:

- Resistance is measured in Ohms (Ω): $1\ \Omega = 1$ V/A
- In the cylindrical wire example, the electric field is uniform everywhere inside the resistor
- For steady currents and uniform conductivity inside a material:

$$\nabla\cdot\mathbf{E} \;=\; \nabla\cdot\left(\frac{\mathbf{J}}{\sigma}\right) \;=\; \frac{1}{\sigma}\nabla\cdot\mathbf{J} \;=\; 0$$

  So there is no unbalanced charge inside the resistor; any excess charge must be on the surface.
  This means we can still use Laplace's equation inside homogenous Ohmic materials carrying steady current (and use the methods of Section 3 to find the potential V).

  There are lots of materials that do not respond to electric fields according to Ohm's law (tungsten filaments in incandescent light bulbs, for example).

Collision model of conductivity

In fact, electrons traveling through a wire collide with atoms in the wire very often, so they can be considered as having an average velocity even though they are always accelerating (with abrupt stops). If the average distance between obstacles is λ and the acceleration is a, the time between collisions is found from

$$\lambda = \tfrac{1}{2}at^2 \quad\Rightarrow\quad t = \sqrt{\frac{2\lambda}{a}}$$

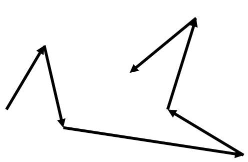
Assuming a full stop at each collision, the average speed is:

$$v_{\text{av}} = \tfrac{1}{2}at = \sqrt{\frac{\lambda a}{2}}$$

So far, this model is wrong! It fails to predict Ohmic conduction, because

$$J \;\propto\; v_{\text{av}} \;\propto\; \sqrt{a} \;\propto\; \sqrt{f}$$

The explanation is that the electrons in the metal are already moving with thermal speeds in random directions. These thermal speeds tend to be much higher than the drift velocity induced by the electric field, so the time between collisions is shorter than given previously. Instead we have



$$t = \frac{\lambda}{v_{\text{thermal}}} \qquad \text{leading us to} \qquad v_{\text{av}} = \tfrac{1}{2}at = \frac{\lambda a}{2v_{\text{thermal}}}$$

In the "classical" regime of thermal physics (e.g., temperature $T \sim 300$ K), we have by the theorem of equipartition of energy:

$$\tfrac{3}{2}k_BT = \tfrac{1}{2}m\,v_{\text{thermal}}^2$$

implying

$$v_{\text{thermal}} \propto \sqrt{T}$$

We can now find the current density, assuming n molecules per unit volume, f free electrons per atom or molecule, and electron charge q and mass m:

$$\mathbf{J} = nfq\mathbf{v}_{av} = nfq\frac{\lambda \mathbf{a}}{2v_{\text{thermal}}} = \frac{nfq\lambda}{2v_{\text{thermal}}}\frac{\mathbf{F}}{m} = \left(\frac{nfq^2\lambda}{2mv_{\text{thermal}}}\right)\mathbf{E}$$

The conductivity, according to this simple model, increases with the number of free electrons per molecule, and decreases with increasing temperature:

$$\sigma = \frac{nfq^2\lambda}{2mv_{\text{thermal}}}$$

Each collision turns electrical energy into heat in the resistor. The work done per unit charge is V, and the charge per unit time is I, so the power dissipated in the resistor is:

$$P = IV = I^2R$$

This is the Joule heating law.

Electromotive force

The current flow in a circuit connected to a battery is uniform all around the circuit. If it were not, then charge would accumulate at some point, creating an electric field that would tend to disperse the charge.
When the circuit is switched on, current almost immediately flows through the whole circuit, even through the individual charges move very slowly. The electric field is established almost instantaneously.

The forces on the charges are therefore the source driving the circuit (e.g., battery, photoelectric cell, or generator) and the electrostatic force. For a unit charge the total force is

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

The net effect of the source on the circuit can be given by the integral of the force per unit charge around the circuit:

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} + \oint \mathbf{E} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

The electric field term dropped out since the curl of **E** is zero.

$\mathcal{E}$ is called the electromotive force (emf) of the circuit.
For an ideal source (like a battery with no internal resistance), the net force of the source on all the charges inside the source is zero (since the conductance is infinite). In the battery, we have

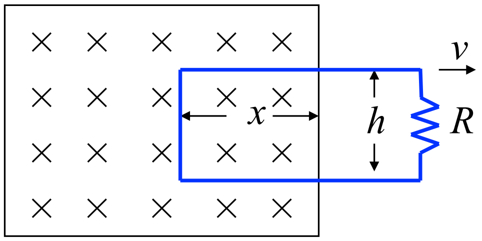
$$\mathbf{E} = -\mathbf{f}_s$$

The potential difference between the terminals a and b (by integrating over the external loop, where the force due to the source is zero) is:

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} = \mathcal{E}$$

The force due to the source makes the charge flow opposite to the direction of current flow in the circuit; the electric field makes the charge flow through the rest of the circuit.
The emf can be interpreted as the work done, per unit charge, by the source.

Motional emf



The simplest form of power generation is a generator, which uses a motional emf. Moving a wire through a magnetic field generates a current. For example, if we have a single rectangular loop, one end of which is in a uniform magnetic field:

When the loop is pulled to the right at speed v, electrons in the left side of the loop experience a magnetic force, which induces a current in the loop.

$$\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh$$

We do the integral around the loop for the emf at a particular instant of time: the only contribution comes from the left side, where $d\mathbf{l}$ is in the vertical direction.

Notice that the person pulling the loop is doing the work, not the magnetic force.
The person exerts a force per unit charge equal to

$$f_{pull} = uB$$

where u is the speed of the charges producing the current (so charges in the left side of the loop actually have a vertical velocity component u and a horizontal velocity component v) .

Change of magnetic flux

For the same example of a current loop, we can express the emf nicely using the flux of the magnetic field through the loop. The flux Φ is defined as B times the area, or more generally:

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a}$$

For our rectangular loop,

$$\Phi = Bhx$$

As the loop moves out of the field, the flux decreases:

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$$

But Bhv is just the emf previously calculated in the loop, and so

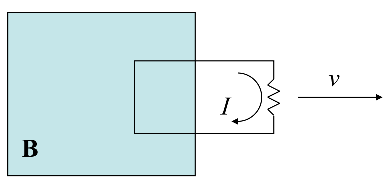
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

It can be proved that this holds for any shape of loop, not just a rectangle.

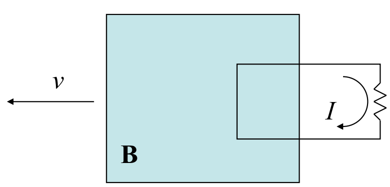
Electromagnetic induction

Faraday's Law

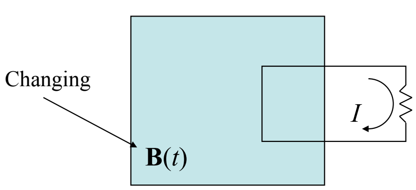
We consider a series of three different experiments that can be performed, leading to rather similar results. These correspond (roughly) to experiments performed by Faraday in the 1830s.



Experiment 1: Pull a loop of wire out of a region of magnetic field; a current flows in the loop.



Experiment 2: Pull the source of the magnetic field away from the loop; a current flows in the loop.



Experiment 3: Leave the wire loop still while changing the magnetic field: a current flows in the loop.

For Experiments 1 and 2

Experiment 1 is just a motional emf, like in the last section. Of course, it isn't much different if the loop or the field is doing the moving, except …

In Experiment 1, charges in the wire experience a magnetic force because they are moving.

In Experiment 2, the field is changing, but the charges aren't moving, so there is no magnetic force; so why are they experiencing a force?

With a proper relativistic treatment, there would be no problem, but this was a puzzle for Faraday. He concluded:

A changing magnetic field induces an electric field.

For Experiment 1, we have already found that

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Faraday found empirically for Experiment 2 that the same result is true:

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

Therefore the electric field is related to the magnetic field as:

$$\oint \mathbf{E} \cdot d\mathbf{l} \quad = \quad -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\boldsymbol{a}$$

This is Faraday's law. In differential form (using Stokes' theorem) it becomes:

$$\nabla \times \mathbf{E} \quad = \quad -\frac{\partial \mathbf{B}}{\partial t}$$

Notice that, in the absence of a changing magnetic field, we get back the static case:

$$\oint \mathbf{E} \cdot d\mathbf{l} \quad = \quad 0$$

For Experiment 3:

The magnetic field in the loop changes for different reasons in this experiment, but we get the same result as in Experiment 2.

For all three experiments, then, we have:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

In the first experiment, it is the Lorentz force law that causes the emf; while in the other two it is the electric field induced by the changing magnetic field.

The strange coincidence that the above physical law holds in difference reference frames led Einstein to look for a fundamental connection between the electric and magnetic fields. This became part of the special theory of relativity.

Lenz's law

In principle, we can find the direction of the current induced in our loop using the right hand rule, once we have defined the positive flux direction.

There is a simpler rule, however, to help (Lenz's law):

Nature acts to oppose a change in flux.

The flux from the induced current will be in the opposite direction to the change in flux which is causing it (it tries to prevent a change in flux).

The induced electric field

Note that Faraday's law has provided us with a new way to produce an electric field. The electrostatic field we discussed in Sections 2—4 was due to electric charges and could be calculated with Coulomb's law (or its equivalent).

We can now see a kind of analogy between Faraday's law

$$\nabla \times \mathbf{E} \quad = \quad -\frac{\partial \mathbf{B}}{\partial t}$$

and Ampere's law:

$$\nabla \times \mathbf{B} \quad = \quad \mu_0 \mathbf{J}$$

Also we have for a pure Faraday field (according to Gauss's law):

$$\nabla \cdot \mathbf{E} = 0$$

while for the magnetic field (as always):

$$\nabla \cdot \mathbf{B} = 0$$

With Ampere's law, it was shown previously that we can find the magnetic field using a loop enclosing the current and applying the integral form:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

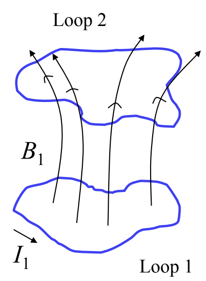
In an analogous way, we can use a loop enclosing the changing magnetic flux to find the Faraday contribution to the electric field:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

If we have a symmetric change in magnetic field, we can use very similar techniques as in the case of Ampere's law.

Inductance

Take the case of two loops of wire placed close to each other. If we run a current through one wire, it causes a magnetic flux through the other.



The Biot-Savart law tells us that the flux through loop 2 is proportional to the current in loop 1:

$$\Phi_2 = M_{21} I_1$$

Here M_{21} is just the proportionality constant; it called the mutual inductance of loop 2 due to loop 1.

We can easily derive a formula for the mutual inductance by using the magnetic vector potential:
We start with the flux written in terms of the vector potential and then use Stokes' theorem:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\boldsymbol{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\boldsymbol{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2$$

The previous result (in Section 5) for the magnetic vector potential can be applied to loop 1:

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{r_c}$$

This gives for the flux in loop 2:

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{r_c} \right) \cdot d\mathbf{l}_2$$

So the mutual inductance is:

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r_c}$$

This is sometimes called Neumann's formula.

We note two things about the mutual inductance:

1) The mutual inductance depends only on the geometry of the two loops, and their relative positions
2) Since it doesn't't matter which line integral we do first, we can switch the two loops without changing the mutual inductance; this means that:

so we can usually drop the subscripts.

A surprising consequence is that, whatever the shape of the loops, the flux through loop 2 due to any current I in loop 1 is the same as the flux through loop 1 if we run the same current I through loop 2.

If we vary the current through one of the loops (say, loop 1), the flux through the other (loop 2) will change, inducing an emf in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

So when you change the current in loop 1, a current flows in loop 2.

Changing the current in loop 1 also induces an emf in loop 1 itself. The flux is again proportional to the current:

$$\Phi = LI$$

We call *L* the self-inductance, or just the inductance, of the loop. The emf induced in that loop is:

$$\mathcal{E} = -L\frac{dI}{dt}$$

Notes:

- The unit of inductance is the Henry (H); 1 H = 1 Vs/A
- Inductance is always positive
- The emf induced always opposes the change in current; it is sometimes referred to as a back emf
- Inductance plays the same sort of role as does the mass in mechanical systems; the larger the value of *L*, the harder it is to change the current.

Energy in magnetic fields

Now that we have looked at changing magnetic fields (through induction), we should be get an expression for the energy stored.

In particular, the work done against the back emf when turning on a current (like the work done charging a capacitor) is recoverable in principle, and can be regarded as energy stored in the magnetic field.

The work per unit time against the back emf is found from:

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

If we turn on the current from 0 to a final value *I*, the total work done is therefore:

$$W = \frac{1}{2}LI^2$$

We would like a more general formula. The product *L I* is the flux through the loop, and the flux is also related to magnetic vector potential **A** by:

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{a} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint_P \mathbf{A}\cdot d\mathbf{l}$$

This gives us:

$$LI = \oint \mathbf{A}\cdot d\mathbf{l}$$

The work is then:

$$W = \frac{1}{2}I\oint \mathbf{A}\cdot d\mathbf{l}$$

Or (after rearranging the vectors):

$$W = \frac{1}{2}\oint (\mathbf{A}\cdot\mathbf{I})dl$$

This can easily be generalized, for example, to volume currents:

$$W = \frac{1}{2}\int_V (\mathbf{A}\cdot\mathbf{J})d\tau$$

We can alternatively rewrite this formula entirely in terms of the magnetic field **B**, just like we found one entirely in terms of the electric field **E** in Section 2. Using Ampere's law to substitute for **J**, we have:

$$W = \frac{1}{2\mu_0}\int \mathbf{A}\cdot(\nabla\times\mathbf{B})d\tau$$

We can move the derivative from **B** to **A** using a product rule (see Griffiths):

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

The energy is now:

$$W = \frac{1}{2\mu_0}\left[\int_V \mathbf{B}\cdot\mathbf{B}\,d\tau - \int_V \nabla\cdot(\mathbf{A}\times\mathbf{B})\,d\tau\right]$$

or

$$W = \frac{1}{2\mu_0}\left[\int_V B^2\,d\tau - \int_S (\mathbf{A}\times\mathbf{B})\cdot d\boldsymbol{a}\right]$$

The integrals must include the entire region occupied by the current, but we can always take a larger volume if we wish.
As we get farther from the current distribution, both **A** and **B** get smaller, so the contribution from the surface integral gets smaller and smaller.
If we make the integral over all space, we are left with only the volume integral.

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

So we can view the energy as being stored either in the current, or in the magnetic field.

Notes:
We couldn't previously calculate the energy of a static magnetic field, because magnetic fields can't do work.
Magnetic fields that are changing, on the other hand, induce electric fields, and we can do work against those.
The formula for the energy stored in a magnetic field is very like the one for the energy in an electric field:

$$W_{\text{elec}} = \frac{1}{2}\int (V\rho)\,d\tau = \frac{\varepsilon_0}{2}\int E^2\,d\tau$$

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})\,d\tau = \frac{1}{2\mu_0}\int B^2\,d\tau$$

Maxwell's Equations

Electrodynamics before Maxwell

So far, we have looked at static electric and magnetic fields, and electric fields induced by changing magnetic fields. This has given us the following set of fundamental relations:

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho$$

Gauss's law

$$\nabla\cdot\mathbf{B} = 0$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

Faraday's law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

Ampere's law

It turns out there is a problem with the generality of Ampere's law! Eventually (~ 1865) it was extended by Maxwell.
The divergence of a curl is always zero (see Chapter 1 of Griffith's). This works fine for Faraday's law:

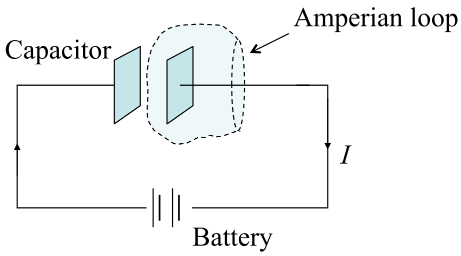
$$\nabla\cdot(\nabla\times\mathbf{E}) = \nabla\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}) = 0$$

But with Ampere's law, we run into a problem:

$$\nabla\cdot(\nabla\times\mathbf{B}) = \mu_0(\nabla\cdot\mathbf{J})$$

The divergence of the current density is zero for steady currents, but it will not in general be zero for currents which change over time. So a contradiction arises!

We can highlight the failure of Ampere's law clearly by looking at a circuit with a charging capacitor:



The current enclosed by our amperian loop should not depend on the surface we choose, but in this case it does:
If we take a flat surface, the current enclosed is I; if we take a bubble which passes between the capacitor plates, the actual enclosed current is zero.

Maxwell's modification for Ampere's law

Maxwell knew that the problem arises because the divergence of **J** is not always zero. However we can apply the continuity equation and Gauss's law to write:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

so

$$\nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) = 0$$

Suppose we now replace **J** in Ampere's law by the quantity in parentheses as above. The modified law will become:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

When **E** is constant, we have our usual Ampere's law; in fact, even when **E** is changing it is usually smaller than the **J** term, and hard to detect.

Maxwell's new term introduces some nice additional symmetry to electromagnetism:
A changing electric field induces a magnetic field, in much the same way that a changing magnetic field induces an electric field.
Following Maxwell, this new term was called the displacement current:

$$\mathbf{J}_{\mathrm{d}} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

It is not a real current in the sense of a flow of real charges, but the extra term has the dimension of current density that gets added to the usual current density in Ampere's law.

Let's go back to the example of a charging capacitor as a test case (see the diagram above):
Now we have two contributions to the curl of the magnetic field in the modified Ampere's law: from any currents crossing whichever surface we choose and from the changing electric field between the capacitor plates.
If the plates are close together, the magnitude of the electric field is

$$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A}$$

This means that between the plates:

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

The new Ampere's law in this case gives us (in its integral form):

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathrm{enc}} + \mu_0 \varepsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\boldsymbol{a}$$

Consider the right-hand side:
Through the flat surface, we have no change in the electric field, but $I_{\mathrm{enc}} = I$.
The bubble surface between the plates has no current flowing through it, but has

$$\varepsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\boldsymbol{a} = I$$

So we now get the same answer no matter which surface we choose.

Maxwell's Equations

Here are the equations of classical electrodynamics:

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho \qquad \text{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's law}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{Ampere's law (with Maxwell's correction)}$$

These tell us how charges produce fields

plus we have the force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

This tells us how fields affect charges

In free space, Maxwell's equations are fairly symmetric:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

When we introduce electric charges, the symmetry is broken. There are free charges (electric monopoles), but no magnetic charge (monopole) has ever been found.

Maxwell's equations in matter

It's useful to get Maxwell's equations in polarized material; we will deal with bound charge and current, so we want to rewrite the equations in terms of only the free charge and current.

We found the bound charge density in terms of the electric polarization in Section 4:

$$\rho_b = -\nabla \cdot \mathbf{P}$$

and in Section 6 the bound current density was obtained:

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

At first sight, we might think that all we need to do is to add ρ_b to ρ and $\mathbf{J}_b$ to $\mathbf{J}$. However, there is an additional current term due to the time dependence of the polarization $\mathbf{P}$. It is called the polarization current $\mathbf{J}_p$ and is given by

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$

The total charge density is then:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$$

and the current density is:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

We can just substitute the above into Maxwell's equation to get the results applicable for a medium with polarization and/or magnetization.

It is convenient to rewrite the some of the equations using the D and H fields defined earlier:

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$$

Maxwell's equation can then be expressed as:

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Note that these are no more general than the original statement of Maxwell's equations, just more convenient when we're dealing with matter.