SECTION 6
Magnetic Fields in Matter

This section (based on Chapter 6 of Griffiths) deals mainly with how magnetic effects are modified in the presence of a magnetic material. The topics are:

- Magnetization
- The field of a magnetized object
- The auxiliary field **H**
- Linear and nonlinear media

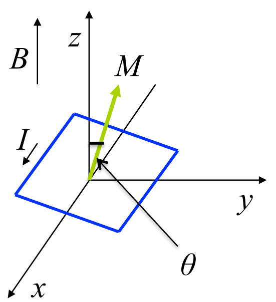
Magnetization

All magnetic field effects are attributed to moving charges (currents). Matter can acquire a magnetization (i.e., a magnetic dipole moment per unit volume) when atomic dipole moments align. There are three main types of materials:
Diamagnets acquire a very weak magnetization opposite an external applied magnetic field, and lose their alignment when the field is removed.
Paramagnets acquire a weak magnetization aligned with an external applied magnetic field, and also lose their magnetization when the field is removed.
Ferromagnets have dipoles which can align with an external magnetic field to produce a much stronger magnetization, and also they retain the magnetization after the field is removed.

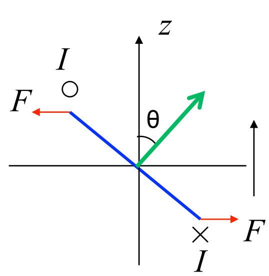
Paramagnets and diamagnets are simpler, because the individual dipole moments do not interact strongly with each other. This results in them being linear (the magnetization is proportional to the applied field); we start with a model of these.
In ferromagnets the individual dipole moments interact strongly (due to quantum mechanics), so they are more complicated and they are nonlinear; we consider them later.

Torques and forces on magnetic dipoles

It is simplest to look first at a rectangular loop carrying a current and consider other shapes later.



The loop is initially assumed in the xy plane with its centre at the origin. Then it is rotated around the x axis so that the normal makes an angle θ with the z axis (and with B).
If the magnetic field B is uniform, the forces on the two slanted sides are equal and opposite (pulling out on the sides of the loop), BUT
the magnetic forces on the other two sides create a torque.



The torque on the loop (tending to rotate it about the x axis) is:

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{x}}$$

where a is the length of the loop on the slanted side. The force on each segment is of magnitude:

$$F = IbB$$

where b is the length of the horizontal sides of the loop. So

$$\mathbf{N} = IabB\sin\theta\,\hat{\mathbf{x}} = mB\sin\theta\,\hat{\mathbf{x}} \quad \text{OR} \quad \mathbf{N} = \mathbf{m}\times\mathbf{B}$$

Although the expression for **N** was derived for a rectangular loop, the equation is valid for any current distribution in a uniform field. (Why? Discuss in class).

In a material the torque tends to line the magnetic dipoles up with the direction of the field (because **N** = 0 when **m** and **B** are parallel vectors). It is the mechanism responsible for paramagnetism.

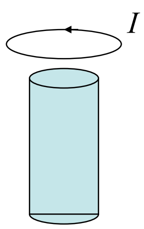
A necessary condition is (obviously) that **m** must be nonzero for the atoms or molecules. Hence those with an odd number of electrons tend to be paramagnets: the spins of the odd electrons line up with the field.

Force on a magnetic dipole

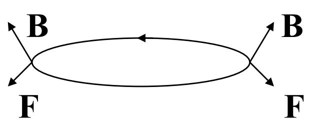
If the field is uniform, the net force on a current loop is zero, because:

$$\mathbf{F} = I\oint(d\mathbf{l}\times\mathbf{B}) = I\left(\oint(d\mathbf{l})\right)\times\mathbf{B} = 0$$

However, if the field is nonuniform, there will generally be a net force on the loop.



For example, if a current loop is placed above a solenoid, field **B** has a radial component because of the fringing fields, so there is a net downward magnetic force.



The magnitude of the net force (vertically down) will be:

$$F = 2\pi I\,R\,B\cos\theta$$

where θ is the angle of **F** with the vertical

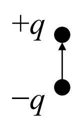
In general, for an infinitesimal loop with dipole moment **m** in a field **B**, it can be shown that the net force is:

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

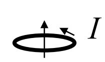
Note that this expression looks just like the force on an electric dipole in a nonuniform electric field:

$$\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})$$

In fact, ideal magnetic and electric dipoles have a very similar behaviour, but the physical origins of the dipoles are quite different.



Real electric dipole



Real magnetic dipole

Effect of a magnetic field on an atomic orbit

In a semi-classical model of atoms, the electrons moves around the nucleus in an orbit, which is like a tiny current loop. In reality, the currents are not simple loops, nor are they steady (since there is only one charge). In practice, however, the electrons move so quickly in their orbits that we can pretend the current is steady, and we can approximate the paths as circles. The equivalent current is:

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$

where the period $T = 2\pi R / v$ with radius = R and speed = v.
The magnetic dipole moments is of magnitude $m = I\pi R^2 = evR/2$,
or vectorially (taking the loop to be in the xy plane):

$$\mathbf{m} = -\frac{1}{2}evR\,\hat{\mathbf{z}}$$

The dipole moment experiences a torque in a magnetic field, but we have already taken account of this as giving rise to paramagnetism. Another effect is that the electron speeds up or slows down, depending on the field direction.

In the absence of a magnetic field, the electric force keeps the electron in its orbit:

$$\frac{e^2}{4\pi\varepsilon_0 R^2} = m_e\frac{v^2}{R}$$

In a magnetic field (which we assume for simplicity to be along z perpendicular to the plane of the orbit), there is an extra Lorentz force acting on the electron. We now have

$$\frac{e^2}{4\pi\varepsilon_0 R^2} + ev'B = m_e \frac{v'^2}{R}$$

where v' is the new speed (and we assume R does not change much).
Now we subtract the above two equations to get:

$$ev'B = m_e \frac{(v'^2 - v^2)}{R}$$

If the speed difference $\Delta v = v' - v$ is relatively small, then

$$(v'^2 - v^2) \quad = \quad (v'+v)(v'-v) \quad \approx \quad 2v\Delta v$$

This gives

$$\Delta v = \frac{eRB}{2m_e}$$

The dipole moment also changes because of the change in speed:

$$\Delta \mathbf{m} = -\frac{1}{2} e(\Delta v) R \hat{\mathbf{z}} = -\frac{e^2 R^2}{4m_e}\mathbf{B}$$

The dipole moment changes in the direction opposite to **B**, and this turns out to be true whatever the direction of **B** relative to the electron loop.

This is the mechanism for diamagnetism. It affects all atoms, but is typically weaker than the paramagnetic effect, so it is only observed when the atoms have a net zero magnetic moment. Only atoms that have an even number of electrons tend to be diamagnetic (and only some of them).
In fact, diamagnetism is actually a quantum-mechanical effect, and so the preceding classical calculation is wrong in detail; still, it illustrates the effect.
The essential property of diamagnetism is that the magnetic dipole moment is in the opposite direction to the applied **B** field, whereas for paramagnetism the magnetic dipole moment is in the same direction as the **B** field.

Magnetization

When the magnetic dipoles in a material align (whether it is with or against the external magnetic field), the material is said to be magnetized. The magnetization vector **M** is a measure of the degree of alignment. By analogy with the definition of polarization **P** in a dielectric medium, we define

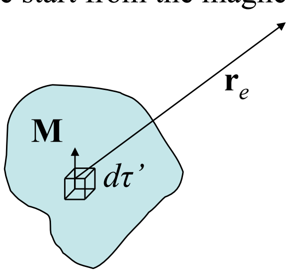
M = Magnetic dipole moment per unit volume

Both paramagnetic and diamagnetic materials experience forces in magnetic fields. In practice these turn out to be very small compared with the ferromagnetic materials to be discussed later (where **M** is much larger)..

The field of a magnetized object

The field from a magnetized object

By analogy with the treatment in the electric field case, we look first at the magnetic field produced by a magnetized object, in the absence of any external field.
We start from the magnetic potential from a single magnetic dipole:



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}_e}{r_e^2}$$

To find the potential due to a magnetized object, we integrate over little volume elements, each with dipole moment $\mathbf{M}\,d\tau'$:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathbf{r}}_e}{r_e^2}\,d\tau'$$

This can be used directly to obtain the correct result for the magnetic potential. However, recalling the electrical case where we replaced the polarized material by bound charges, we will follow a similar method to calculate $\mathbf{A}(\mathbf{r})$. As before (Section 4), we use the fact that:

$$\nabla'\left(\frac{1}{r_e}\right) = \frac{\hat{\mathbf{r}}_e}{r_e^2}$$

with the prime indicating differentiation with respect to $\mathbf{r}_e$, and then we get

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\left[\mathbf{M}(\mathbf{r}')\times\nabla'\left(\frac{1}{r_e}\right)\right]d\tau'$$

Integrating by parts:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\int\frac{1}{r_e}\left[\nabla'\times\mathbf{M}(\mathbf{r}')\right]d\tau' - \int\nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{r_e}\right)d\tau'\right\}$$

We can eventually express the second term on the right as a surface integral (it involves some more vector mathematics and then the use of the divergence theorem). The results is:

$$\mathbf{A}(\mathbf{r}) \quad = \quad \frac{\mu_0}{4\pi}\int\frac{1}{r_e}\left[\nabla'\times\mathbf{M}(\mathbf{r}')\right]d\tau' \quad + \quad \frac{\mu_0}{4\pi}\oint\frac{1}{r_e}\left[\mathbf{M}(\mathbf{r}')\times d\boldsymbol{a}'\right]$$

Potential of a volume current   Potential of a surface current

$$\mathbf{J}_b = \nabla\times\mathbf{M} \qquad\qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

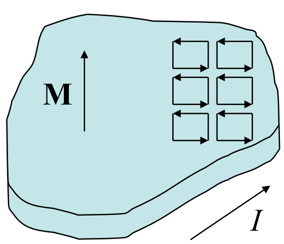
With these definitions, we have:

$$\mathbf{A}(\mathbf{r}) \quad = \quad \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_b(\mathbf{r}')}{r_e}d\tau' \quad + \quad \frac{\mu_0}{4\pi}\oint\frac{\mathbf{K}_b(\mathbf{r}')}{r_e}da'$$

So, instead of integrating over the magnetization, we can instead calculate the bound surface and volume currents and calculate the field directly from them.
This is a direct parallel to the bound surface and volume charges we used in the electrostatic case for the electric field of a polarized object.

Bound currents

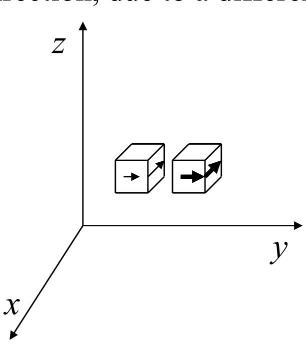
Like bound charges, bound currents are real, and not just a mathematical construct.



Surface bound currents are a result of the little current loops in a magnetized object failing to cancel at the boundary: the net effect is like a current flowing around the boundary of the object.
The surface current must always be normal to both the magnetization of the material and to the surface, hence (as found before):

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

Volume bound currents occur when the magnetization is nonuniform. Suppose we look at the net current in the x direction, due to a difference in magnetization in the y and z directions:



At the interface between the two current loops, the net current in the x direction is:

$$I_x = [M_z(y+dy) - M_z(y)] = \frac{\partial M_z}{\partial y}dydz$$

This corresponds to a current density contribution of:

$$(J_b)_x = \frac{\partial M_z}{\partial y}$$

Similarly, any change in magnetization in the z direction would produce a corresponding term in the net x current, giving

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}$$

The right side is just the x component of a curl. Extending this to 3 dimensions, we get the expected result:

$$\mathbf{J}_b = \nabla \times \mathbf{M}$$

The auxiliary field H

Ampere's law in magnetized materials

Now that we have the field due to the bound currents, we can move on to get the full field, due to the free currents and the magnetization of matter. The total current (with bound and free terms) is:

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f$$

We can write Ampere's law as:

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_b + \mathbf{J}_f = (\nabla \times \mathbf{M}) + \mathbf{J}_f$$

or, collecting the two curl terms:

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f$$

It is convenient to define a new field **H** for the term in the parentheses, so the result becomes

$$\mathbf{H} = \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \right) \qquad \text{OR} \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$$

Ampere's law can then be written:

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

In integral form it is:

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f-enc}$$

Therefore we can write Ampere's law in terms of the free current only. Also, when symmetry allows it, we can calculate **H** from Ampere's law.

Comments on H

H is called the auxiliary field by Griffiths; in many other books it is called the magnetic intensity. Often it is loosely referred to the magnetic field, but this term is properly reserved for **B**.

While Ampere's law in terms of **H** looks similar to Ampere's law in terms of **B**, they can't be used the same way, because we need to take account of the divergence as well as the curl. While the divergence of **B** is always 0, the divergence of **H** is not necessarily 0, and

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$$

Boundary conditions for H

From the previous boundary conditions found for **B**, we can easily deduce the corresponding boundary conditions for **H** as:

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$$

for the component perpendicular to the surface, and

$$H^{\parallel}_{\text{above}} - H^{\parallel}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

for the component along the surface (note that it is only affected by the free current in the parallel component).

Linear and nonlinear media

Magnetic susceptibility

For linear magnetic materials (usually paramagnets and diamagnets) , the magnetization is proportional to the applied magnetic field.

By analogy with the electric field case (where we used **P** proportional to **E** to define an electrical susceptibility), we might expect that **M** proportional to **B** would be used to define a magnetic susceptibility. This is **NOT** what happens. Instead, the convention is to use the **H** field and define

$$\mathbf{M} = \chi_m \mathbf{H}$$

The dimensionless proportionality constant χ_m is called the magnetic susceptibilty.
For paramagnetic and diamagnetic materials, χ_m is always much less than one: usually around 10^{-5}.

Magnetic Permeability

For linear magnetic media:

$$\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$$

This means that the magnetic field **B** is also proportional to **H**, so we can write

$$\mathbf{B} = \mu \mathbf{H}$$

where

$$\mu = \mu_0(1+\chi_m)$$

is called the permeability of the material. Most linear materials have a permeability close to the permeability of free space.

Even though **B** and **H** are proportional inside linear media, we cannot conclude that the divergence of **H** always vanishes. To see this, we consider:

$$0 \quad = \quad \nabla\cdot\mathbf{B} \quad = \quad \nabla\cdot(\mu\mathbf{H}) \quad = \quad \mu\nabla\cdot\mathbf{H}+\mathbf{H}\cdot(\nabla\mu)$$

This implies that

$$\nabla\cdot\mathbf{H} \quad = \quad -\frac{1}{\mu}\mathbf{H}\cdot(\nabla\mu)$$

This will vanish inside the magnetic material (if μ is constant), but on the boundary it will be very large because there is a discontinuity of μ.

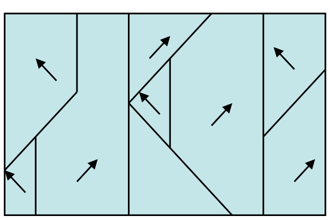
The volume bound current in a homogenous linear material is proportional to the free current density:

$$\mathbf{J}_b \quad = \quad \nabla\times\mathbf{M} \quad = \quad \nabla\times(\chi_m\mathbf{H}) \quad = \quad \chi_m\nabla\times\mathbf{H} \quad = \quad \chi_m\mathbf{J}_f$$

Hence, if no current flows through the material, all bound currents will be at the surface.

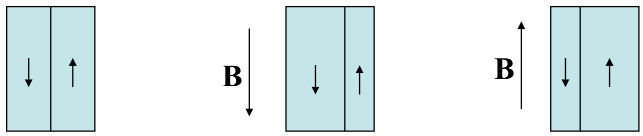
Ferromagnetism

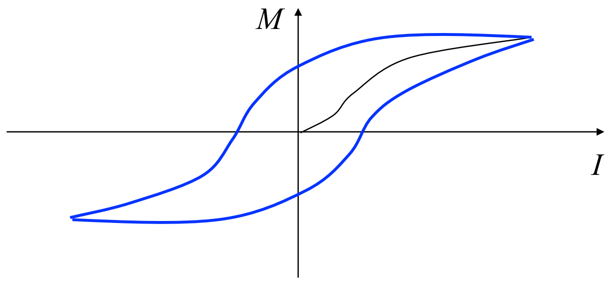
Like in a paramagnet, the dipoles in a ferromagnet tend to align parallel to an external magnetic field. However, a major difference is that in a ferromagnet the magnetic dipoles on one atom interact strongly with the dipoles on neighbouring atoms (giving an extra tendency to line up). By contrast, in a paramagnet these interactions are negligible. The extra interactions in ferromagnets are mainly quantum-mechanical and are called exchange interactions.
Because of these differences, the magnetization **M** in ferromagnets can be much greater (by several orders of magnitude) than in paramagnets. Also, unlike paramagnets, the tendency for each dipole to align with its neighbours means that ferromagnets keep their magnetizations when the external field is removed.



In an unmagnetized piece of iron, the alignments among dipoles occur in small domains: all the dipoles are aligned in the domain, but each domain is randomly oriented with respect to the others.

When a magnetic field is applied to an unmagnetized piece of ferromagnetic material, the domains which are magnetized in the direction of the field grow at the expense of their neighbours magnetized in the opposite direction.
When the **B** field is removed, the domains do not shift back all the way to their original state (i.e., the process is irreversible).





The magnetization of a ferromagnet (such as iron) depends not only on the applied magnetic field at that instant, but also on the history. If we place a piece of iron in a coil, and apply a current, the magnetization traces out a hysteresis loop as shown above.

Notes: The magnetic field from the magnetized ferromagnetic material is much greater than the magnetic field applied. This is because typically

$$\chi_m >> 1$$

If the temperature is increased, random thermal motions tend to disorder the magnetic alignment. For ferromagnetic materials, there is a particular temperature (called the Curie point), above which the material becomes paramagnetic instead of ferromagnetic (e.g., for iron, this is 770ºC).

Magnetostatic Problems: Summary for sections 5—6

Usually, we have a current distribution and want to find the magnetic field (sometimes we find the potential first).

System	Method for solving
• Many line currents	• Biot-Savart Law and superposition
• Continuous current distribution with cylindrical, plane, or solenoid symmetry	• Ampere's Law
• Extended distribution with other symmetry	• Integrate over current distribution to get vector potential
• Want potential far from current distribution	• Multipole expansion (vector potential of dipole)
• Magnetized material	• Find bound currents and calculate field
• Linear material in magnetic field	• Find field **H** from Ampere's law, calculate **B**.