SECTION 3
Special Techniques

This section (based on Chapter 3 of Griffiths) extends the concepts of the previous section by covering some special techniques that are useful in electrostatics. The topics are:

- Laplace's equation
- The method of images
- Separation of variables
- Multipole expansion

Laplace's Equation

In Section 2, we used Coulomb's Law as a way to find the electric field from a given charge distribution:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{volume}} \frac{\rho(\mathbf{r}')}{r_e^2}\hat{\boldsymbol{r}}_e d\tau'$$

Although we can use this in simple situations, the integral rapidly becomes intractable for many situations. The potential integral is usually easier to solve, being a scalar function – but this can still be a difficult integral to solve.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{r_e} d\tau'$$

Sometimes, it is easier to look at the differential form of the potential equation (Poisson's equation):

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

If we have information about the boundaries (i.e., we are given boundary conditions for V), we can solve this differential equation.

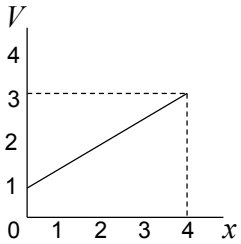
A simpler situation occurs if we only want to find the potential in a region where there are no charges (naturally there must be charge somewhere, otherwise the problem would be trivial; there is just an absence of charge in the region we are interested in).
In this case, the potential is given by Laplace's equation: $\nabla^2 V = 0$
In Cartesian coordinates it is:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

(and other coordinate systems will be considered later).

Laplace's equation in one dimension

We start with a very simple case (not corresponding to any useful real situation): $\frac{\partial^2 V}{\partial x^2} = 0$



Integrating twice gives

$$V(x) = mx + b$$

where m and b are constants of integration.
If we are given some boundary conditions, like $V(0) = 1$ and $V(4) = 3$, we can find m and b.

We can deduce some basic facts from the one-dimensional solution which will carry over to higher dimensions as well:

1. The value $V(x)$ at any x is just the average of $V(x+a)$ and $V(x-a)$, for any interval a. The function never does anything exciting between two points.

$$V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$$

2. Because of this, there are no local maxima or minima in the solutions to Laplace's equation. There can only be maxima or minima on the boundary.

Laplace's equation in two dimensions

A more realistic case is when two dimensions have variations. Laplace's equation is:

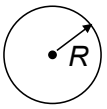
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Since this is a partial differential equation rather than an ordinary differential equation, there is no completely general solution, like in one dimension (1D).

Still, we can state some general properties of the solution by analogy with the 1D case:

1. The value of V at any point (x, y) is the average of all the points around the chosen point. For example, if we take a circle of radius R around the point, the average value on the circle is equal to the value at the centre.

$$V(\mathbf{r}) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl$$



2. The function V has no local maxima or minima: all of the extrema occur on the boundaries.

Laplace's equation in three dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

For the general 3D case, we again do not have an explicit solution of the partial differential equation.
It can be shown that we still have the same general properties for the solutions as in 1D and 2D:

1. The value of V at a given point is the average of the values around that point:

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da$$

2. Again, V cannot have local maxima or minima: these can only occur on the boundaries.

Boundary conditions and uniqueness

In solving Laplace's equation for V, the question arises: **If we find a solution that satisfies the equation and its boundary conditions, is that solution unique?**

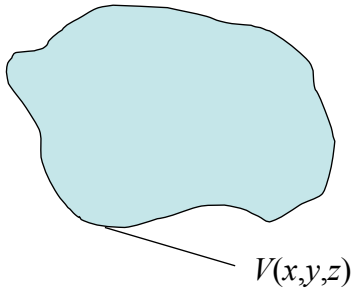
In 1D it's easy to see the boundary conditions that will give us a unique solution:

- The values of V at each end of the line, or alternatively
- One value of V and its derivative.

Note that giving the derivative at 2 points would be redundant (because the gradient is the same everywhere) and does not give information to determine a unique solution.

In 2D or 3D it is not so obvious what the conditions are for uniqueness (or if there are any at all) – in fact, there are two useful results telling us about uniqueness of the solutions.

First Uniqueness Theorem



Suppose we want to solve for the potential V inside some arbitrary volume, given a boundary condition that specifies $V(x,y,z)$ on the surface of this volume.
The first uniqueness theorem states that the solution inside the volume must be unique.

Proof: Assume that for a given set of boundary conditions, there are two correct solutions:

$$\nabla^2 V_1 = 0 \qquad \text{and} \qquad \nabla^2 V_2 = 0$$

If we can prove that V_1 must be equal to V_2, then we will have a contradiction, meaning that there is only one possible solution.

We define the difference between the 2 functions by $V_3 = V_1 - V_2$

It follows that V_3 must also obey Laplace's equation, since $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$. In addition, V_3 must be zero on the boundary, since the two solutions agree there.

We had the result that solutions to Laplace's equation have no local maxima or minima, so both the maximum and absolute minimum of V_3 must be zero. This means that V_3 is identically zero, so we have proved the result.

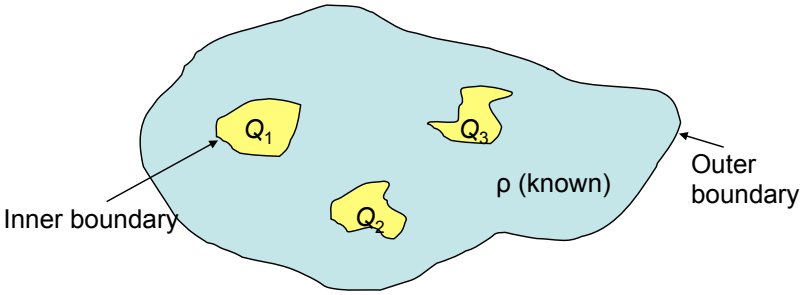
We can extend the proof to show that *the uniqueness theorem still applies to regions with some charge density:*

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0} \quad \text{and} \quad V_3 = V_1 - V_2$$

Obviously, V_3 still satisfies Laplace's equation because $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0$

Again, the two solutions must be the same on the boundary, so the difference V_3 must be zero there and hence everywhere, and so $V_1 = V_2$

Second Uniqueness Theorem



This applies to situations where we have a region of known charge density ρ interspersed by some conductors. Each conductor *i* can have a net charge Q_i on it, but the distribution over the conductor is not known.

The outer boundary can be either another conducting surface or just infinity.

This theorem states the electric field is uniquely determined (which means also that there's only one way the charge can distribute itself over the conductors).

Proof: As before, we assume there are two electric fields ($\mathbf{E}_1$ and $\mathbf{E}_2$) that satisfy the problem. For each, in the volume between the conductors, we use Gauss's Law:

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\varepsilon_0}\rho \quad \text{and} \quad \nabla \cdot \mathbf{E}_2 = \frac{1}{\varepsilon_0}\rho$$

Over a surface S_i enclosing each conductor *i*:

$$\oint_{S_i} \mathbf{E}_1 \cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0} Q_i \quad \text{and} \quad \oint_{S_i} \mathbf{E}_2 \cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0} Q_i$$

Also, at the outer boundary *B*:

$$\oint_B \mathbf{E}_1 \cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0} Q_{tot} \quad \text{and} \quad \oint_B \mathbf{E}_2 \cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0} Q_{tot}$$

Like we did in the last case, we look at the difference $\mathbf{E}_3 = \mathbf{E}_1 - \mathbf{E}_2$ and we quickly conclude that

$\nabla \cdot \mathbf{E}_3 = 0$ between conductors, and $\oint \mathbf{E}_3 \cdot d\boldsymbol{a} = 0$ across all surfaces

An additional piece of information is that the surface of each conductor is an equipotential, so V_3 must be a constant over each conductor.

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3 (\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2$$

where we used the product rule for differentiation and then the fact that the divergence of $\mathbf{E}_3$ is 0 and the gradient of potential is related to the electric field.

Now , we can apply the divergence theorem to the left side (integrating over the volume between conductors):

$$\oint_{all\ S_i} (V_3\mathbf{E}_3)\cdot d\boldsymbol{a} \;=\; \int_{vol} \nabla\cdot(V_3\mathbf{E}_3)\,d\tau \;=\; -\int_{vol} (E_3)^2\,d\tau$$

BUT the surface integral on the left must vanish, because the potential is constant over each surface and so we can bring it outside the integral, and we know from a previous result that the integral of the electric field over any boundary is zero, so:

$$\int_{vol} (E_3)^2\,d\tau \;=\; 0$$

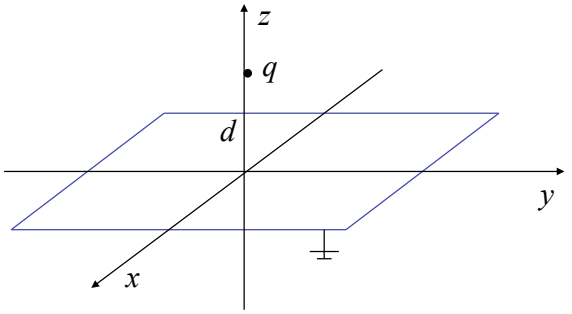
Since the integrand here is never negative, the only way for it to vanish is that it is zero everywhere. This means that $\mathbf{E}_1 = \mathbf{E}_2$ as required.

The method of images

Image method for infinite flat conductor

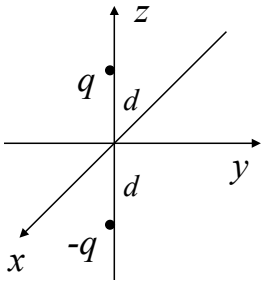
Suppose we consider a point charge q held a distance d above a flat, infinite, grounded conducting plate.

We want to find the potential $V(x,y,z)$ in the region $z > 0$ above the plate, which will include the potential of the positive charge and from any negative charges induced on the plate below the charge.



The "direct" approaches are either to solve Poisson's equation in the half-plane $z > 0$, with boundary condition $V = 0$ at $z = 0$ and $V \to 0$ far from the charge or to solve first for induced charge distribution on the plate and then find V.

A much easier approach is to use the image method (which employs symmetry).



In this approach we use the first uniqueness theorem, which tells us that if we find another potential which has the right behaviour for $z > 0$ and satisfies the same boundary conditions, then it's the correct potential, no matter how we found it.

With that in mind, we find the potential for a different charge configuration: two charges placed above and below the $z = 0$ plane as shown.

This is exactly the same physical system in the upper half plane (so Poisson's equation is the same there, and also the plane $z = 0$ is an equipotential with $V = 0$.
So the solution must be the same in the physical region.

Now it's easy to work out the distance from any point (x, y, z) to charge q at $(0, 0, d)$ and image charge $-q$ at $(0, 0, -d)$. The potential due to the 2 charges is

$$V(x,y,z) \;=\; \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

From this, we could work out the electric field from the gradient of V.

Induced surface charge

We might want to know how the induced charge is distributed on the conducting plate. An earlier expression for the surface charge was:

$$\sigma \;=\; \varepsilon_0 E_{\text{normal}} \;=\; -\varepsilon_0 \frac{\partial V}{\partial z}$$

where in the last step we use the fact that the normal to the plate is the z direction.

We can calculate the derivative:

$$\frac{\partial V}{\partial z} \;=\; \frac{1}{4\pi\varepsilon_0}\left[\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

which leads to the following result for the surface charge:

$$\sigma(x,y) \;=\; \frac{-qd}{2\pi[x^2+y^2+d^2]^{3/2}}$$

We can also find the total induced charge on the plate by integrating over the total surface. It's slightly easier in polar coordinates:

$$Q \;=\; \int \sigma\, da \qquad \text{and} \qquad \sigma(r) \;=\; \frac{-qd}{2\pi(r^2+d^2)^{3/2}}$$

Therefore $$Q \;=\; \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}}\, r\, d\phi\, dr \;=\; \left.\frac{qd}{(r^2+d^2)^{1/2}}\right|_0^{\infty} \;=\; -q$$

So the total induced charge is $-q$ (as expected).

Force and energy in the image system

The negative charge induced on the plane attracts the point charge. The potential at the position of q is the same as for the two-charge problem; so the field must also be the same and therefore the force must be the same.

$$\mathbf{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

For the energy, however, we have to be more careful. The energy for the image system of two point charges is:

$$W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)}$$

However, for the real system of a single charge near a conducting plane, we need to multiply this energy by a factor of one half in order to get the correct result, which is:

$$W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(4d)}$$

To see why this is so, we look at the energy stored in the fields. In each case this is given by

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau$$

In the case with the conducting plate, the field only extends over the upper half-space ($z>0$), whereas for the image system it is symmetrically over all space. Hence the one-half factor.

Separation of variables

Separation of variables in Cartesian coordinates

Suppose we have the potential V or the charge density specified over the boundaries of some region, and we want to solve for V in the interior.

A method that sometimes works is to look for separable solutions of the form (taking Cartesian coordinates):

$$V(x,y,z) \;=\; X(x)\,Y(y)\,Z(z)$$

i.e., the potential is written as the product of three functions, where each is a function of a single coordinate only. There are no guarantees that this works, but we can try it and see.

In a region with no charge, the potential obeys Laplace's equation:

$$\frac{\partial^2[X(x)Y(y)Z(z)]}{\partial x^2}+\frac{\partial^2[X(x)Y(y)Z(z)]}{\partial y^2}+\frac{\partial^2[X(x)Y(y)Z(z)]}{\partial z^2} = 0$$

Doing the partial differentiation, this becomes:

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2}+X(x)Z(z)\frac{d^2Y(y)}{dy^2}+X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0$$

Now divide through by V:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$

The only way that this can be possible is if each term on the left (one is a function of x only, one of y only, and one of z only) is a constant. We call them c_1, c_2, and c_3, respectively, and they add up to zero ($c_1 + c_2 + c_3 = 0$).

Then we have: $\frac{d^2X}{dx^2} = c_1X$ and similarly for the other parts.

Each part is now an ordinary differential equation and can be solved in the usual way.

The solution of the above equation depends on whether c_1 is positive or negative (which we usually know from the boundary conditions).

If c_1 is positive (and we denote $c_1 = \alpha^2$, with α real), then the solutions have the form

$$X(x) = Ae^{-\alpha x}+Be^{\alpha x}$$

If c_1 is negative (and we denote $c_1 = -\beta^2$, with β real), then the solutions have the form

$$X(x) = C\sin(\beta x)+D\cos(\beta x)$$

Special case of two dimensions

Sometimes only 2 of the 3 dimensions are involved, e.g., if only x and y variables are involved and there is no z dependence. Then

$$V(x,y) = X(x)\,Y(y)$$

and the separate equations for $X(x)$ and $Y(y)$ involve constants such that $c_1 + c_2 = 0$, so one is positive and the other is negative.

Taking the case where $c_1 = \alpha^2 > 0$ and $c_2 < 0$, we have solutions like

$$V(x,y) \quad \left[Ae^{-\alpha x}+Be^{\alpha x}\right]\left[C\sin(\beta y)+D\cos(\beta y)\right]$$

The rest of the calculation involves applying the boundary conditions.

Case of spherical coordinates

If the system has spherical symmetry, it may be easier to use separation of variables in spherical coordinates:

$$V(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

Laplace's equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2V}{\partial\phi^2} = 0$$

For applications that are important for this course, we will assume there is no ϕ dependence of V. The equation reduces to:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

and we look for solutions of the form: $V(r,\theta,\phi) = R(r)\Theta(\theta)$

Substituting this, simplifying, and dividing through by V gives:

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)+\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right) = 0$$

Since the first term only depends on r and the second only on θ, both must be constants.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=c \quad \text{and} \quad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-c$$

Next it can be proved (but not here) that the θ equation has unphysical divergences at $\theta = 0$ and π, UNLESS c is an integer taking the form

$$c=\ell(\ell+1) \quad \text{where} \quad \ell=0,1,2,3,\cdots$$

Now we solve the separate equations:

First, there is the radial equation which becomes $$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \ell(\ell+1)R$$

To solve we might guess a power law like $R = r^n$. Substitution gives $n(n+1)r^n = \ell(\ell+1)r^n$ which has 2 solutions as $n=\ell$ and $n=-(\ell+1)$

The general radial solution is $$R(r) = Ar^{\ell}+\frac{B}{r^{\ell+1}}$$

Next, the angular equation is more complicated: $$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -\ell(\ell+1)$$

The solutions to this equation are special functions known as Legendre polynomials. They are polynomials in powers of cosθ:

$$\Theta(\theta) = P_{\ell}(\cos\theta)$$

There are general formulas to find these polynomials, but the first few are:

$$P_0(\cos\theta)=1\ ,\quad P_1(\cos\theta)=\cos\theta\ ,\quad P_2(\cos\theta)=(3\cos^2\theta-1)/2\ ,\quad P_3(\cos\theta)=(5\cos^3\theta-3\cos\theta)/2\ .$$

The basic solution for V is therefore

$$V(r,\theta) = \left(Ar^{\ell}+\frac{B}{r^{\ell+1}}\right)P_{\ell}(\cos\theta) \qquad \text{for any } \ell$$

In general, there might be several values of l (depending on the boundary conditions), so

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_{\ell}r^{\ell}+\frac{B_{\ell}}{r^{\ell+1}}\right)P_{\ell}(\cos\theta)$$

Case of cylindrical coordinates

We now look for a solution with separation of variables like

$$V(s,\phi,z) = S(s)\Phi(\phi)Z(z)$$

For simplicity, we will take the case where there is no z dependence. Laplace's equation is:

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

We substitute in $V(s, \phi) = S(s)\,\Phi(\phi)$ and rearrange to get:

$$\frac{s}{S}\frac{d}{ds}\left(s\frac{dS}{ds}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

Again, each term must be a constant. Consider first the angular term: it will either have sin and cos solutions (if its constant is negative) or it will have real exponential solutions (if its constant is positive).
Only the first case is physically possible because the solution must repeat (be periodic) for intervals of 2π for the angle ϕ.

We therefore take the constant to be negative:

$$\frac{d^2\Phi}{d\phi^2} = -k^2\Phi$$

and the solutions are of the form:

$$\Phi(\phi) \;=\; A\sin k\phi + B\cos k\phi \qquad \text{where } k \text{ is an integer} = 0, 1, 2, 3, \ldots$$

The radial equation is now:

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right) \;=\; k^2 S$$

We can try to solve by guessing; take as a trial solution: $S = s^n$ where n is unknown

Substituting it into the differential equation gives:

$$n^2 s^{n-1} \;=\; k^2 s^{n-1}$$

Therefore n is k or $-k$, and the solution is of the form

$$S(s) \;=\; C s^k + D s^{-k}$$

The only exception is when $k = 0$, which is a special case.

When $k = 0$ the differential equation is:

$$s\frac{d}{ds}\left(s\frac{dS}{ds}\right) = 0 \qquad \text{which gives} \qquad \frac{dS}{ds} = \frac{C}{s}$$

So $S \;=\; C\ln s + D$

We can also check on the angular part when $k = 0$: $\dfrac{d^2\Phi}{d\phi^2} = 0$

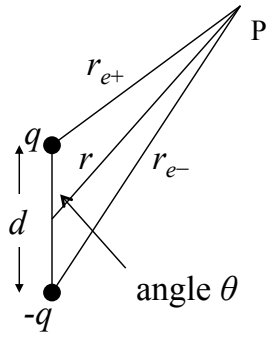
Although this has the solution $\Phi = B\,\phi + A$ we are forced to put $B = 0$, leaving only a constant (remember that Φ must be periodic).

The general solution becomes:

$$V(s,\phi) \;=\; a_0 + b_0 \ln s + \sum_{k=1}^{\infty}[s^k(a_k\cos k\phi + b_k \sin k\phi) + s^{-k}(c_k\cos k\phi + d_k \sin k\phi)]$$

Multipole expansion

Electric dipoles



If we are very far away from a charge distribution, eventually the field can be approximated as if it is due to a point charge Q, where Q is the total charge in the distribution.

But what if the net $Q = 0$? We start by considering a dipole of 2 charges q and $-q$ at a distance d apart.

The potential at P is:

$$V(\mathbf{r}) \;=\; \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_{e+}} - \frac{q}{r_{e-}}\right)$$

where $r_{e\pm}^2 \;=\; r^2 \mp rd\cos\theta + (d/2)^2 \;\approx\; r^2\left(1 \mp \dfrac{d}{r}\cos\theta\right)$ and we assume large distances $r >> d$.

Then, using the Binomial expansion

$$\frac{1}{r_{e\pm}} \;\approx\; \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \;\approx\; \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

So we get $\dfrac{1}{r_{e+}} - \dfrac{1}{r_{e-}} \;\approx\; \dfrac{d\cos\theta}{r^2}$

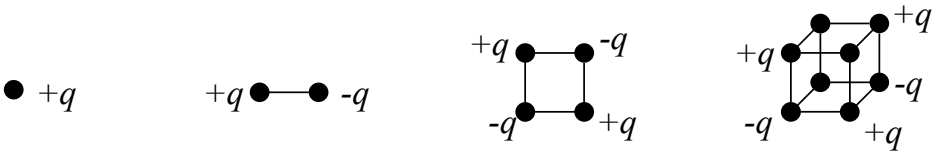
and the approximate result for the potential becomes

$$V(\mathbf{r}) \;=\; \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}$$

Note that the potential of a dipole falls of as $1/r^2$, instead of $1/r$ as in the case of a point charge.

Series of multipoles

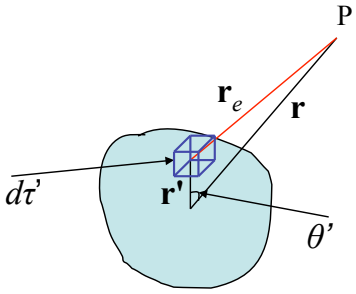
Monopole Dipole Quadrupole Octopole



$V \sim 1/r$ $V \sim 1/r^2$ $V \sim 1/r^3$ $V \sim 1/r^4$ and so on.

General multipole expansion

Suppose we have a general charge distribution ρ and we want to calculate the potential V at a distant point P:



$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r_e}\rho(\mathbf{r}')d\tau'$$

where $r_e^2 = r^2 - 2rr'\cos\theta' + (r')^2 = r^2\left[1 - 2\left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\right]$

Therefore $\frac{1}{r_e} = \frac{1}{r}\left[1 - 2\left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\right]^{-1/2}$

After a lot of math (doing substitutions and using the Binomial expansion) without making any approximations, it can eventually be proved that

$$\frac{1}{r_e} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

where we have the same Legendre polynomials as before. Now we can express V at a given position as:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\,\rho(\mathbf{r}')d\tau' \qquad \text{or}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\,\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots\right]$$

monopole potential *dipole potential* *quadrupole potential*

The first term dominates at large distances if the total charge Q is nonzero:

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \text{where} \quad Q = \int \rho(\mathbf{r}')d\tau'$$

If the total charge is 0, the next term (dipole term) dominates:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')d\tau'$$

We can reorganize this expression using: $r'\cos\theta' = \hat{\mathbf{r}}\cdot\mathbf{r}'$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')d\tau'$$

Finally, we can rewrite the result by defining the dipole moment as: $\mathbf{p} = \int \mathbf{r}'\rho(\mathbf{r}')d\tau'$ which gives us:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

For n point charges the general definition of $\mathbf{p}$ becomes

$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}_i'$$

In the special case of two opposite charges (like before) separated by distance d:

$$\mathbf{p} = q\mathbf{r}_+' - q\mathbf{r}_-' = q(\mathbf{r}_+' - \mathbf{r}_-') = q\mathbf{d}$$

Notes:

- V_{dip} is a good approximation far away from a physical dipole (or when d tends to 0); but closer to the dipole there are higher order correction terms.
- Dipole moments add as vectors, so the total dipole moment of a quadrupole is 0.
- The dipole moment, as well as all higher moments, typically depends on the choice of origin of the coordinate system (except in the special case of 2 charges as above).

Electric field of a dipole

We can use the results for the potential to work out the electric field. We take a dipole, located at the origin and pointing in the z direction.

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

We now take the gradient of V to get the electric field (choosing to use spherical polar coordinates):

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \qquad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} \qquad E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

The total electric field can be written as

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

Note that the field components go like $1/r^3$. The above results could also be worked out directly for a dipole of 2 charges q and $-q$.