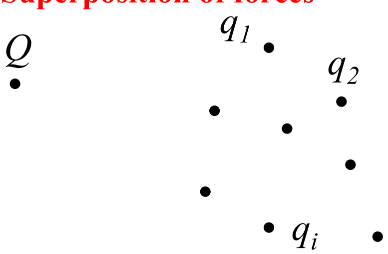
SECTION 2
Electrostatics

This section, based on Chapter 2 of Griffiths, covers effects of electric fields and forces in static (time-independent) situations. The topics are:

- Electric field
- Gauss's Law
- Electric potential
- Work and energy
- Conductors

Electric field

Superposition of forces



Suppose we have some static arrangement of source charges (q_1, q_2, q_3 and so on), and we want to be able to determine the force on another charge (a "test" charge Q).

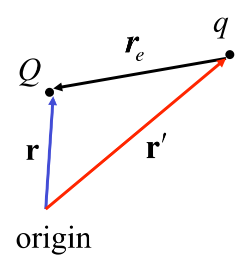
The principle of superposition states that the interaction of two charges is independent of any other charges present.

To find the force from q_1, we can ignore the other charges. Then we can do the same for each of the other charges, and do a vector sum of the forces.

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots$$

Hence the problem is reduced to solving for the force between any one source charge and the test charge.

Coulomb's Law



Now we need the force that that any source charge q exerts on our test charge Q.
From experiments, it's been determined that

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_e^2}\hat{\boldsymbol{r}}_e$$

The constant ε_0 is called the permittivity of free space. In SI units:

$$\varepsilon_0 = 8.85\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$

The separation vector between q and Q is as defined earlier: $\boldsymbol{r}_e = \mathbf{r} - \mathbf{r}'$

The force is attractive if q and Q have opposite signs, and repulsive if they have the same sign.

Electric Field

Now going back to the case where there are many source charges, the total force is:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1Q}{r_{e1}^2}\hat{\boldsymbol{r}}_{e1} + \frac{q_2Q}{r_{e2}^2}\hat{\boldsymbol{r}}_{e2} + \cdots\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{e1}^2}\hat{\boldsymbol{r}}_{e1} + \frac{q_2}{r_{e2}^2}\hat{\boldsymbol{r}}_{e2} + \cdots\right)$$

We can now define the electric field vector $\mathbf{E}$ so that $\mathbf{F} = Q\,\mathbf{E}$, so that

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_{ei}^2}\hat{\boldsymbol{r}}_{ei}$$

for a system of n charges.
The electric field is a function of position, but does not depend on the test charge. It is a real entity, because we will show that it carries energy and momentum.

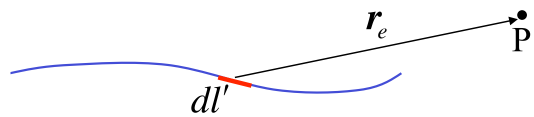
Continuous charge distributions

By changing the sum to an integral, we can calculate the field due to a continuous distribution of charge:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r_e^2}\hat{\boldsymbol{r}}_e$$

Here dq is an element of charge and its value will depend on what shape we are integrating over.
There are 3 cases of interest:

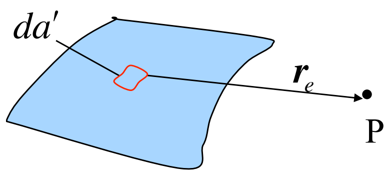
Line charge
(λ per unit length)



$$dq \rightarrow \lambda\, dl'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{line}} \frac{\lambda(\mathbf{r}')}{r_e^2}\hat{\boldsymbol{r}}_e dl'$$

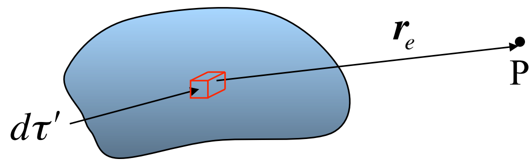
Surface charge
(σ per unit area)



$$dq \rightarrow \sigma\, da'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{surface}} \frac{\sigma(\mathbf{r}')}{r_e^2}\hat{\boldsymbol{r}}_e da'$$

Volume charge
(ρ per unit volume)

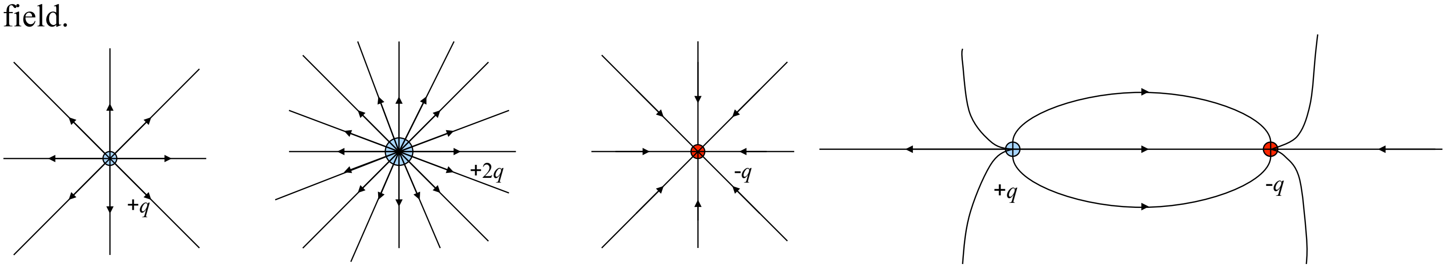


$$dq \rightarrow \rho\, d\tau'$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{volume}} \frac{\rho(\mathbf{r}')}{r_e^2}\hat{\boldsymbol{r}}_e d\tau'$$

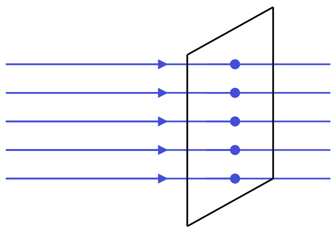
Gauss's Law

Electric field lines are a useful way to represent the electric field schematically. They originate at positive charges, terminate on negative charges, and can extend to infinity. The denser the lines, the higher the electric field.

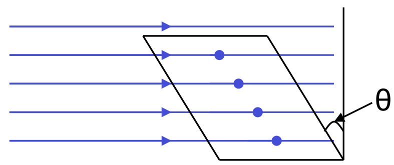


Electric Flux

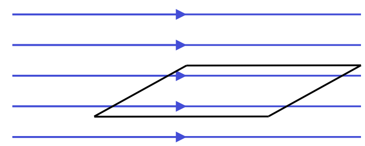
By considering any surface in an electric field, we can calculate the flux, which is a measure of the number of field lines crossing a surface (of area A held at right angles):



Flux = EA

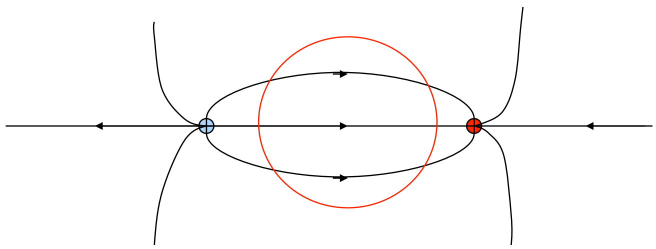


Flux = $EA \cos\theta$

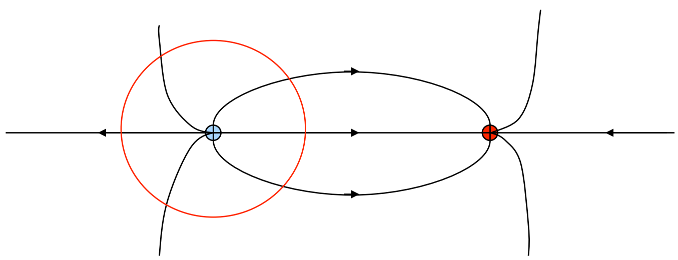


Flux = 0

In general, $\Phi_E = \int_{\text{surface}} E \cdot d\boldsymbol{a}$



Schematically it is expected that a closed surface which does not enclose any charge has an equal number of lines entering and leaving (zero net flux). For example (see opposite):



If, on the other hand, a surface encloses a charge, there will be a net number of lines entering or leaving, i.e., a net flux that is proportional to the charge.

From these qualitative arguments, we can conclude that the flux through a closed surface depends only on the amount of charge enclosed by the surface.

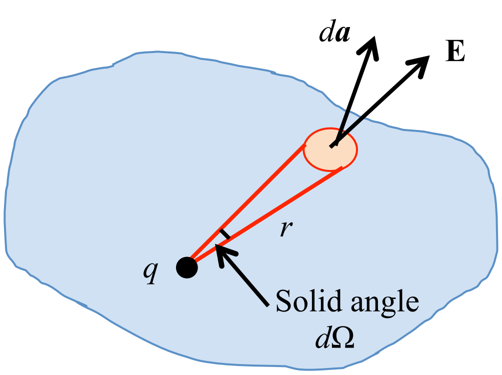
Gauss's Law

As a simple example, the flux through a sphere, with radius r, centred on a charge q placed at the origin is:

$$\oint_{\text{sphere}} \mathbf{E} \cdot d\boldsymbol{a} = \frac{q}{4\pi\varepsilon_0 r^2} 4\pi r^2 = \frac{1}{\varepsilon_0} q$$

Although the math is easier with a sphere, it works for <u>any</u> shape, and the charge can be positioned anywhere.

A general proof (using solid angles) goes as follows:



Consider a point charge q anywhere within an arbitrary shaped volume, and denote by $d\boldsymbol{a}$ a vector element of surface area (meaning its magnitude is the area da and its direction is the outward normal to the surface). The direction of the electric field $\mathbf{E}$ is radially outward from the charge q.

Flux through the element of area

$$= \mathbf{E} \cdot d\boldsymbol{a} = \frac{q\,\hat{\mathbf{r}} \cdot d\boldsymbol{a}}{4\pi\varepsilon_0 r^2} = \frac{q\,d\Omega}{4\pi\varepsilon_0}$$

where the last step follows from the definition of solid angle.

Therefore the total flux for the closed surface is

$$\int_{\text{surface}} \mathbf{E} \cdot d\boldsymbol{a} = \frac{q}{4\pi\varepsilon_0} \int_{\text{all directions}} d\Omega = \frac{q}{\varepsilon_0}$$

In general for n charges enclosed, by the principle of superposition

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i \qquad \text{and so} \qquad \oint \mathbf{E}\cdot d\boldsymbol{a} = \sum_{i=1}^{n}\left(\oint \mathbf{E}_i\cdot d\boldsymbol{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\varepsilon_0}q_i\right)$$

We can now summarize this as follows.

Gauss's Law says that the flux of **E** through a closed surface is:

$$\oint_{\text{surface}} \mathbf{E}\cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0}Q_{\text{enc}}$$

where the total charge enclosed is Q_{enc}. (The charges outside make no difference to the result.)

This is really just another way of stating Coulomb's Law (in an integral format), but it is a very useful result for cases which have spherical, cylindrical and plane symmetry (because the integrals then become easy).

Gauss's Law in differential form

We can change the integral form of Gauss's Law to an equivalent differential form (which is neater, though the integral form is more generally useful).

First, we apply the divergence theorem to the left-hand side:

$$\oint_{\text{surface}} \mathbf{E}\cdot d\boldsymbol{a} = \int_{\text{volume}} \nabla\cdot\mathbf{E}\,d\tau$$

Then, if we have a continuous charge distribution, the right-hand side becomes:

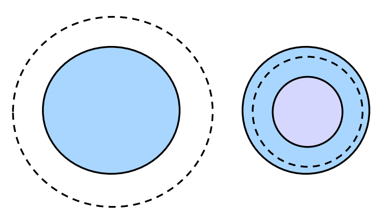
$$\frac{1}{\varepsilon_0}Q_{\text{enc}} = \int_{\text{volume}} \frac{1}{\varepsilon_0}\rho\,d\tau$$

Since this is true no matter what volume we choose, it follows that:

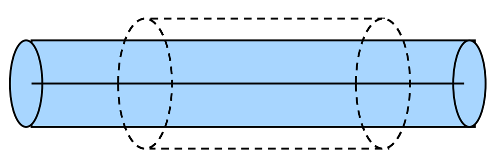
$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho$$

Gauss's Law applications

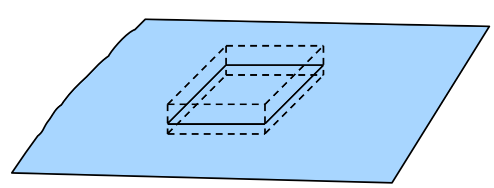
Spherical symmetry:
To find the field at radius r, choose a surface with that radius.



Cylindrical symmetry:
Again, choose a surface (this time cylindrical) with radius s.



Plane symmetry:
Choose a flat "pillbox", straddling the surface.



The electric field must be constant on that surface and perpendicular to it, so that the surface integral becomes a multiplication.
Examples will be done in classes.

Electric potential

Curl of E

Suppose we want to find the curl of the field **E** due to a single point charge q at the origin:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

We could do this directly, but another way is to calculate the integral of **E** along some line:

In spherical coordinates: $$d\mathbf{l} = dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\theta} + r\sin\theta d\phi\,\hat{\phi}$$

So: $$\int_a^b \mathbf{E}\cdot d\mathbf{l} = \frac{q}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{r}\right]_a^b = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

For a closed path, we have $a = b$, and $$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

Finally, by Stokes' theorem (see Section 1): $$\nabla\times\mathbf{E} = 0$$

Electric Potential

We saw in Section 1 that if the curl of a vector field is 0, it can be represented as the gradient of a scalar. Since we know the curl of an electrostatic field is always 0, we can represent it as the gradient of a scalar field, called the electric potential V.

Being a scalar, the potential is usually much easier to work with than the field vector, which has components. It is often easier to work out the potential due to a charge distribution first, and then calculate the field using:

$$\mathbf{E} = -\nabla V$$

Notes:

- ❑ The origin of V is arbitrary because we can add any constant to V with changing **E**.
- ❑ Therefore we choose whatever is convenient. Unless the charge distribution is infinite, we usually take $V = 0$ at infinity.
- ❑ The electric potential, like the electric field, obeys the superposition principle.
- ❑ The electric potential has units of Volts.

Poisson's Equation and Laplace's Equation

The fundamental electric field equations,

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho \quad \text{and} \quad \nabla\times\mathbf{E} = 0 \quad ,$$

can be rewritten in terms of the potential V.

For the divergence of electric field:

$$\nabla\cdot\mathbf{E} = \nabla\cdot(-\nabla V) = -\nabla^2 V$$

So, from Gauss's Law: $$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$ This is called Poisson's equation.

In the special case of $\rho = 0$ (zero charge density in the region), we have:

$$\nabla^2 V = 0$$ This is known as Laplace's equation.

We will look at solutions to Poisson's and Laplace's equation in Section 3.

In many problems we are given a charge distribution and we want to find the electric potential V. We can (in principle) integrate Poisson's equation to get V.

First, for a point charge q at the origin:

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_\infty^r \frac{q}{r'^2}dr' = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r'}\right]_\infty^r = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

In general, for a point charge at any position:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_e}$$

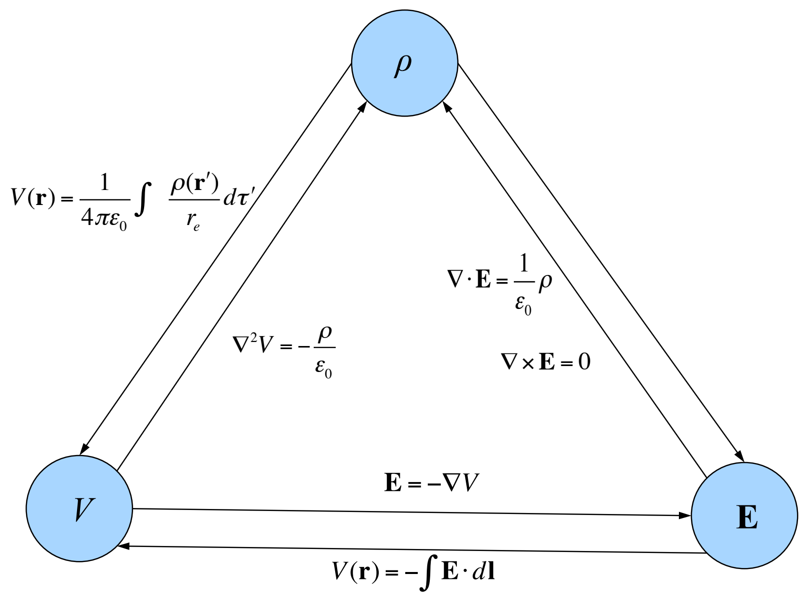
And we can write down the potential for many charges using the superposition principle.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_{ei}}$$

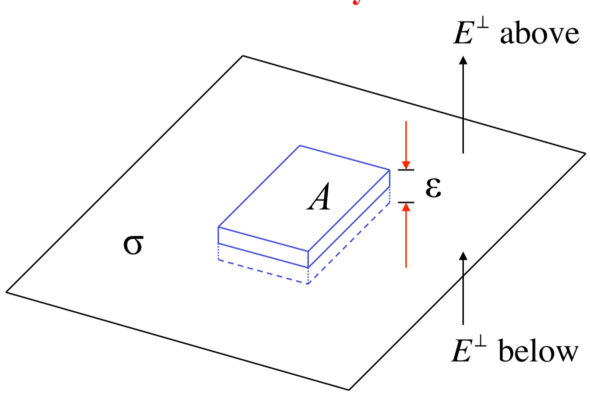
For a continuous (volume) distribution of charge, we can integrate as:

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{r_e}d\tau'$$

Summary of relationships



Discontinuities and boundary conditions



The electric field changes discontinuously at a sheet of charge. It will be useful to determine how the field changes at this boundary.

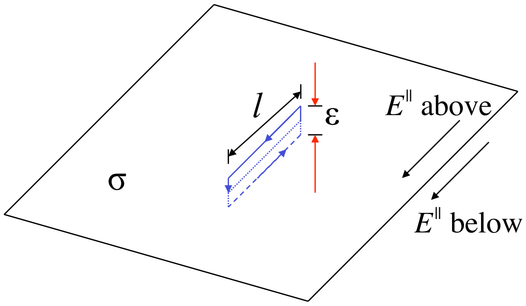
Using Gauss's Law on the little pillbox, we get:

$$\oint_S \mathbf{E}\cdot d\boldsymbol{a} = \frac{1}{\varepsilon_0}Q_{\text{enc}} = \frac{1}{\varepsilon_0}\sigma A$$

If we make the thickness ε extremely small, the sides of the box do not contribute:

$$E^\perp_{above} - E^\perp_{below} = \frac{1}{\varepsilon_0}\sigma$$

So we have a boundary condition that the perpendicular component of the field is discontinuous by an amount σ / ε_0 at any boundary that has some surface charge.



Next we look at the tangential component of **E**. Take a small loop very close to the surface.

We know that

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

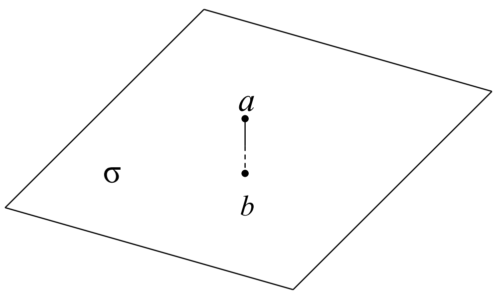
from Stokes' Law.

Now we can make the vertical length ε very small (compared to l), so that the sides do not contribute to the integral. This means that the top and bottom must cancel, so

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

We can combine these two boundary conditions in vector form as:

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{1}{\varepsilon_0}\sigma\hat{\mathbf{n}}$$



Knowing the change in electric field, we can deduce the change in potential V:

$$V_{\text{above}} - V_{\text{below}} = -\int_a^b \mathbf{E}\cdot d\mathbf{l}$$

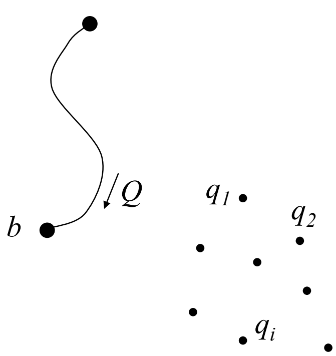
If we make point a very close to b, the integral will shrink to 0, so:

$$V_{\text{above}} = V_{\text{below}}$$

Of course, we know that the gradient of V still has a discontinuity (because the gradient of V is just $-\mathbf{E}$ and we have shown that the perpendicular component of **E** has a discontinuity).

Work and energy

Work to move a charge



What is the work needed to move charge Q from point a to point b in the presence of any distribution of charges?

At any point on the path, the electric force on Q is

$$\mathbf{F} = Q\mathbf{E}$$

The work done (by you) in moving the charge is therefore:

$$W = -\int_a^b \mathbf{F}\cdot d\mathbf{l} = -Q\int_a^b \mathbf{E}\cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\boldsymbol{a})]$$

Obviously, the work is independent of path: this is why we can call the electrostatic force conservative.

The expression for the work done can be rearranged as:

$$V(\mathbf{b}) - V(\boldsymbol{a}) = \frac{W}{Q}$$

We see that the difference in electric potential is the work per unit charge to move a particle between the points.

To bring a charge from far away, the work is:

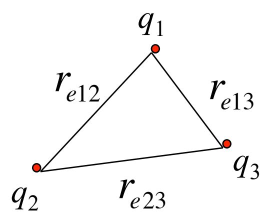
$$W = Q[V(\mathbf{r}) - V(\infty)]$$

So, if we choose to set $V = 0$ at infinity,

$$W = QV(\mathbf{r})$$

Hence potential is just potential energy per unit charge (like the field is force per unit charge).

Energy of a charge distribution



Now we can look at the work needed to assemble a distribution of point charges. Imagine doing this by bringing them in from infinity one by one.

No work is needed to bring the first charge from infinity ($W_1 = 0$).

The work to bring in the 2nd charge is: $$W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{r_{e12}} \right)$$

And then a 3rd charge: $$W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{r_{e13}} + \frac{q_2}{r_{e23}} \right)$$

And then a 4th charge: $$W_4 = \frac{1}{4\pi\varepsilon_0} q_4 \left(\frac{q_1}{r_{e14}} + \frac{q_2}{r_{e24}} + \frac{q_3}{r_{e34}} \right)$$, assuming the case of 4 charges in total.

We add these separate amounts up to get the total work done to bring these four charges from infinity:

$$W = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 q_2}{r_{e12}} + \frac{q_1 q_3}{r_{e13}} + \frac{q_1 q_4}{r_{e14}} + \frac{q_2 q_3}{r_{e23}} + \frac{q_2 q_4}{r_{e24}} + \frac{q_3 q_4}{r_{e34}} \right)$$

For n charges, this generalizes to become:

$$W = \frac{1}{2} \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j=1 (i \neq j)}^{n} \frac{q_i q_j}{r_{eij}}$$

The factor of ½ is to avoid double counting each term, and obviously we need to exclude terms with i and j the same.

This is useful in this form, but it can also be written another way by taking out the sum over i:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j=1 (i \neq j)}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{r_{eij}} \right)$$

The term in the brackets is just the electric potential V at the position of the i^{th} charge. We can now rewrite the expression for the work as:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

Energy of a continuous distribution

We can find the energy of a continuous charge distribution by changing the sum in the last expression to an integral:

$$W = \frac{1}{2} \int \rho V \, d\tau$$

This is itself often a useful expression, but there's another useful integral form in terms of the electric field.

First, we use Gauss's Law to get the charge density in terms of **E**: $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$

This implies $$W = \frac{\varepsilon_0}{2} \int (\nabla \cdot \mathbf{E}) V \, d\tau$$

We can next transfer the derivative from the electric field to the electric potential, by using integration by parts:

$$W = \frac{\varepsilon_0}{2}\left[-\int \mathbf{E}\cdot\nabla V\, d\tau + \int V\mathbf{E}\cdot d\boldsymbol{a}\right] = \frac{\varepsilon_0}{2}\left[\int \mathbf{E}\cdot\mathbf{E}\, d\tau + \int V\mathbf{E}\cdot d\boldsymbol{a}\right]$$

where in the last step we used the definition $\mathbf{E} = -\nabla V$.

We now have a volume integral over a volume that must be large enough to contain all the charge distribution, plus a surface integral (surrounding that volume). However, if we extend the volume to all space, the second integral goes to 0 because **E** and V go to zero fast enough. So we find the energy as:

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\, d\tau$$

The energy can therefore be viewed as stored in the electric field.

Note that the superposition principle does not hold for energy. (This is because the energy of two systems has cross terms in the fields):

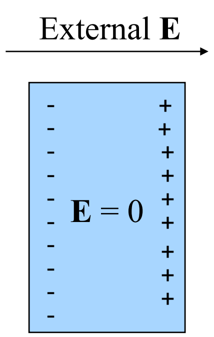
$$W_{\text{tot}} = \frac{\varepsilon_0}{2}\int E_{\text{tot}}^2\, d\tau = \frac{\varepsilon_0}{2}\int (\mathbf{E}_1 + \mathbf{E}_2)^2\, d\tau$$

$$= \frac{\varepsilon_0}{2}\int (E_1^2 + E_2^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2)\, d\tau = W_1 + W_2 + \varepsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\, d\tau$$

Conductors

Basic properties of conductors

In a conductor, the charges move around freely. In metals, the electrons carry charge: in liquids, positive and negative ions do. For now, we will look at perfect or ideal conductors, which have unlimited free charge and no resistance. We want to know what happens to the electric field, the charge density and the potential.



Electric field inside a conductor: Since charge can move freely, the charges inside a conductor will arrange themselves to cancel any external field, so the electric field inside a conductor is always 0.

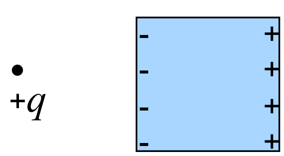
Charge density inside a conductor: Since the electric field inside the conductor is zero, the net charge must also be zero (equal amounts of positive and negative charge).

Charge density at the surface of a conductor: If the conductor is in an electric field, positive and negative charges separate until the field is cancelled. These unbalanced charges must reside on the surface only.

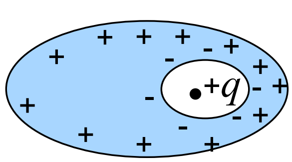
Electric potential in a conductor: Since the electric field is zero inside the conductor, the change in potential is also zero. This means that the conductor is an equipotential.

Electric field just outside a conductor: The field cancels inside the conductor; outside the field may be nonzero. The external field must, however, be perpendicular to the surface at all points (if there was a component along the surface, charge would flow to neutralize it).

Induced charge on a conductor



A conductor in the presence of a charge will be attracted to the charge because of the charge induced on the surface of the conductor.



If we now place a charge q inside a cavity in a conductor, the inner surface will acquire the opposite charge ($-q$) to cancel the field inside the conductor. The outer surface will have charge q, so that the total net charge of the conductor plus the cavity is q.

If there is no charge in the cavity, the field in the cavity must be zero.

Surface charge on conductors

We calculated earlier the change in electric field at a surface charge. Since the field inside a conductor is zero, it follows that the field just <u>outside</u> must be:

$$\mathbf{E} = \frac{1}{\varepsilon_0}\sigma\hat{\mathbf{n}}$$

We can also write the surface charge in terms of the potential:

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$ where the derivative is in the direction of the normal.

Force on a conductor

The force on a charge is just the charge times the electric field.

In the case of a surface charge, the field is discontinuous: the force on the surface charge is the local surface charge times the average of the field inside and outside the surface. For a unit area:

$$\mathbf{f} = \sigma\mathbf{E}_{\text{average}} = \tfrac{1}{2}\sigma(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

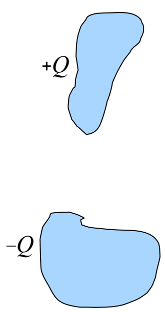
For a conductor, then, the force per unit area is:

$$\mathbf{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{\mathbf{n}}$$

which pulls the conductor into the field. We can also look at this as a pressure, and we can rewrite it in terms of the field just outside the conductor:

$$P = \frac{\varepsilon_0}{2}E^2$$

Capacitors



Imagine two conductors (of any shape): one has charge $+Q$, the other charge $-Q$. We can look at the potential difference between them (each, of course, must be an equipotential):

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E}\cdot d\mathbf{l}$$

From Coulomb's Law, we know that the field is proportional to Q; from the above equation, we see that V must also be proportional to Q.

The proportionality constant is called the capacitance C: $$C = \frac{Q}{V}$$

Work and capacitors

We want the work done to charge a capacitor from charge 0 initially to charge Q finally. Imagine doing this gradually, so that at some intermediate stage the charge is q (which means the voltage difference is q/C).

The work to transfer a small extra amount of charge (call it dq) from the positive plate of a capacitor to the negative plate is found from the equation for work:

$$dW = \text{voltage} \times dq = \frac{q}{C}dq$$

We can integrate from $q = 0$ to $q = Q$ to get the total work:

$$W = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C}$$

Sometimes it is more useful to express the result in terms of voltage: $$W = \frac{1}{2}CV^2$$