Physics 3300A October 10, 2012

Midterm 1

Name: ______________________________

Time allowed: 45 minutes
The use of simple calculators (with no stored information) is permitted. A formula sheet is provided; no other aids are permitted.
Answer the 3 problem questions and the 3 short-answer questions; use the space on this test paper to provide your answers. The marking scheme (out of a total of 40) is indicated.

Problem 1 (10 marks)

(a) Calculate the divergence and curl of the following vector function **F** in Cartesian coordinates:

$$\mathbf{F}(x,y,z) = xy\,\hat{\mathbf{x}} - 2xz\,\hat{\mathbf{y}} + 3\left(\frac{y}{z}\right)\hat{\mathbf{z}}$$

Is it possible for the above vector to represent an electric field **E** in electrostatics? Give the reason for your answer. [*Hint*: think what you know about the curl of an electric field].

(b) Suppose the electric field vector $\mathbf{E}(x,y,z)$ in a box-shaped region of space defined by

$$0 < x < a, \quad 0 < y < b, \quad 0 < z < c\ ,$$

where a, b and c are positive constants, is specified by $\mathbf{E} = 2yz\,\hat{\mathbf{z}}$. Write down the integral expression for the total energy W stored in the electric field in this volume and evaluate W.

(a) $\nabla \cdot \underline{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}\left(\frac{3y}{z}\right) = y + 0 - \frac{3y}{z^2} = y\left(1 - \frac{3}{z^2}\right)$

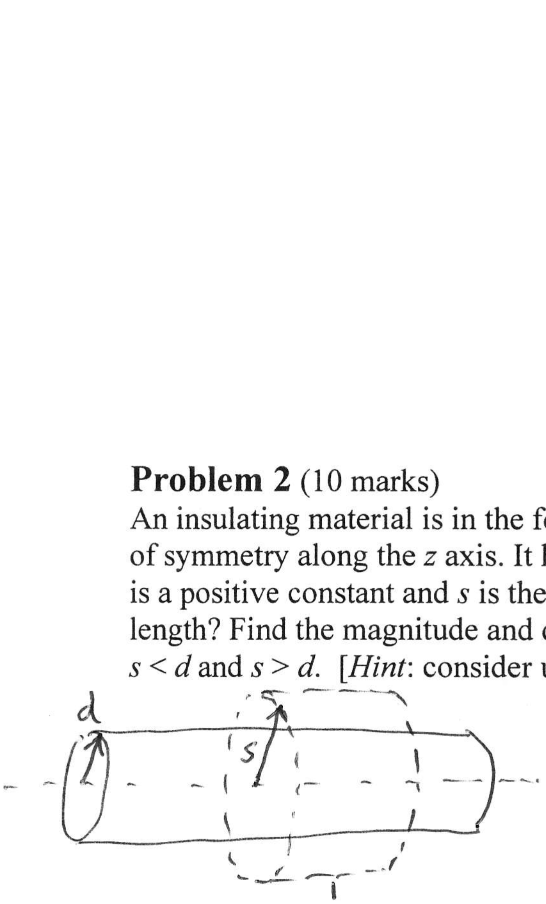
$\nabla \times \underline{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -2xz & 3y/z \end{vmatrix} = \left(2x + \frac{3}{z}\right)\hat{x} - (x + 2z)\hat{z}$

The vector cannot represent an $\underline{E}$ field (because $\nabla \times \underline{E} = 0$ everywhere).

(b) $W = \frac{1}{2}\varepsilon_0 \int E^2 d\tau = \frac{1}{2}\varepsilon_0 \iiint 4y^2 z^2\, dx\,dy\,dz$

$= 2\varepsilon_0 \int_0^a dx \int_0^b y^2 dy \int_0^c z^2 dz = \frac{2}{9}\varepsilon_0\, a b^3 c^3$

Problem 2 (10 marks)
An insulating material is in the form of an infinitely long solid cylinder of radius *d* with its axis of symmetry along the *z* axis. It has a charge density $\rho = -ks^2$ per unit volume for $s < d$, where *k* is a positive constant and *s* is the distance from the *z* axis. What is the total charge per unit length? Find the magnitude and direction of the electric field as a function of *s*, for both cases of $s < d$ and $s > d$. [*Hint*: consider using Gauss's law for a chosen closed surface and volume].



For a unit length

$$\lambda = \int \rho \, da = -k \int_0^d s^2 \, 2\pi s \, ds$$

$$= -\frac{\pi k d^4}{2}$$

By symmetry, electric field must be in radial direction, and since $\lambda < 0$ the electric field must be <u>radially inwards</u> for all s

For Gauss's law, use a cylindrical volume of unit length.

If $s > d$, all the charge ($= \lambda$) is enclosed.

$$\therefore \int \underline{E} \cdot \underline{da} = \frac{1}{\varepsilon_0} Q_{enc} \Rightarrow E . 2\pi s . 1 = \frac{1}{\varepsilon_0}\left(-\frac{\pi d^4 k}{2}\right)$$

$$\therefore E = -\frac{kd^4}{4\varepsilon_0 s} \quad \text{for } s > d$$

(−ve means direction inwards)

Now take $s < d$ and draw an inner cylinder.

Charge enclosed (per unit length) is now $-\frac{\pi k s^4}{2}$

From Gauss's law, $2\pi s E = -\frac{\pi k s^4}{2\varepsilon_0}$

$$\therefore E = -\frac{ks^3}{4\varepsilon_0} \quad \text{for } s < d$$

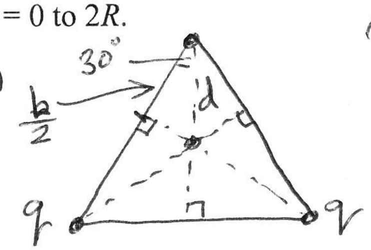
(inwards direction)

Problem 3 (10 marks)

(a) Two identical point charges q are placed at a distance b apart. A third point charge $3q$ is now brought from infinity and placed so that the three charges form an equilateral triangle. How much work was done on the third charge to achieve this? Write down the total potential at the mid-point of the triangle (equidistant from all the charges).

(b) An ideal conductor is in the form of a sphere of radius R and it carries a total charge q. Write down the magnitude of the electric field and potential (i) inside the conductor and (ii) at the surface just outside the conductor. Sketch the potential $V(r)$ as a function of radial distance from $r = 0$ to $2R$.

(a)



Potential at 3rd point due to original two charges

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{b} + \frac{1}{4\pi\varepsilon_0}\frac{q}{b} = \frac{q}{2\pi\varepsilon_0 b}$$

$\therefore$ Work to bring a $3q$ charge to this point

$$= 3q\left(\frac{q}{2\pi\varepsilon_0 b}\right) = \frac{3q^2}{2\pi\varepsilon_0 b}$$

Distance d to midpoint must satisfy $\frac{b}{2} = d\cos 30^\circ \Rightarrow d = \frac{b}{\sqrt{3}}$

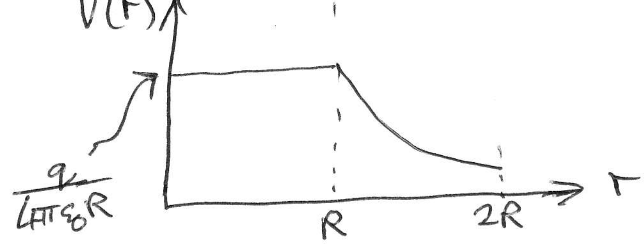
$$\therefore \text{Total } V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{d} + \frac{q}{d} + \frac{3q}{d}\right) = \frac{5\sqrt{3}\,q}{4\pi\varepsilon_0 b}$$

(b) (i) $E = 0$ inside the conductor, and

V = constant inside the conductor $= \frac{q}{4\pi\varepsilon_0 R}$ to match potential just outside

(ii) $E = \frac{q}{4\pi\varepsilon_0 R^2}$ just outside, and $V = \frac{q}{4\pi\varepsilon_0 R}$

For $r > R$, $V = \frac{q}{4\pi\varepsilon_0 r}$



Short question 1 (3 marks)

Two identical spherical shells are place a short distance apart. A total charge Q is now spread evenly over the surface of one of the spheres, requiring an amount of work W_0. Would you expect the amount of work to spread the same amount of charge Q evenly over the second sphere to be less than W_0, equal to W_0, or greater than W_0? Explain.

We expect more work to be done in the 2nd case (because work has to be done against repulsion effects due to the first charge).

$\therefore W > W_0$

Short question 2 (3 marks)

Consider a Gaussian surface that encloses a volume with no charge inside it. Mark each of the following statements as true (T) or false (F).

F a) The electric field on the surface must be zero.

F b) No electric field lines may pass through the surface.

T c) The number of field lines entering the surface is equal to the number of lines leaving the surface.

Short question 3 (4 marks)

Find the value of each of the following integrals involving Dirac delta functions:

$$\int_0^3 x^2\,\delta(x+1)\,dx \quad ; \quad \int_{-1}^{1}(x-1)\,\delta(2-3x)\,dx$$

$\int_0^3 x^2 \delta(x+1) = 0$, because the delta function is at $x=-1$ which is outside the integration range

$\int_{-1}^{1}(x-1)\delta(2-3x)\,dx = \frac{1}{3}\int_{-1}^{1}(x-1)\delta(x-\frac{2}{3})\,dx = -\frac{1}{9}$

(e.g. make a substitution such as $p=2-3x$ and $dp=-3dx$)