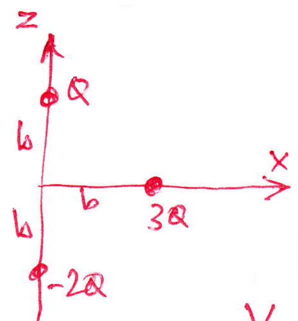
Midterm 2

Name: ______________________________

Time allowed: 45 minutes
The use of a simple calculator (with no stored information) is permitted. A formula sheet is provided; no other aids are permitted.
Answer the 3 problem questions and the 3 short-answer questions; use the space on this test paper to provide your answers. The marking scheme (out of a total of 40) is indicated.

Problem 1 (10 marks)

Point charges of Q, $3Q$, and $-2Q$ are placed at positions with (x, y, z) coordinates given by $(0, 0, b)$, $(b, 0, 0)$, and $(0, 0, -b)$, respectively, where b is a positive length. Using the formula sheet for the multipole expansion (or otherwise), obtain expressions for the monopole and dipole contributions to the scalar potential $V(\mathbf{r})$ at a distant point $\mathbf{r}$ from the charges. What are the effective monopole moment and dipole moment of this combination of charges?



Angle θ' is $0, 90°$, and $180°$ for charges $Q, 3Q, -2Q$ resp.

For monopole term,

$$V_{mono}(\underline{r}) = \frac{1}{4\pi\varepsilon_0 r}\int \rho(\underline{r}')d\tau' = \frac{1}{4\pi\varepsilon_0 r}\sum_i q_i = \underline{\frac{2Q}{4\pi\varepsilon_0 r}}$$

For dipole term,

$$V_{dip}(\underline{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\int r'\cos\theta' \rho(\underline{r}')d\tau' = \frac{1}{4\pi\varepsilon_0 r^2}\sum_i r_i q_i \cos\theta_i$$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\{bQ.1 + b(-2Q)(-1)\} = \frac{3bQ}{4\pi\varepsilon_0 r^2}$$

Effective monopole moment $= \sum_i q_i = \underline{2Q}$

Effective dipole moment is found from $\underline{p} = \sum_i q_i \underline{r}_i$

$$\rightarrow \{Qb - b(-2Q)\}\hat{\underline{z}} + \{b(3Q)\}\hat{\underline{x}}$$

$$= \underline{3bQ\hat{\underline{z}} + 3bQ\hat{\underline{x}}}$$

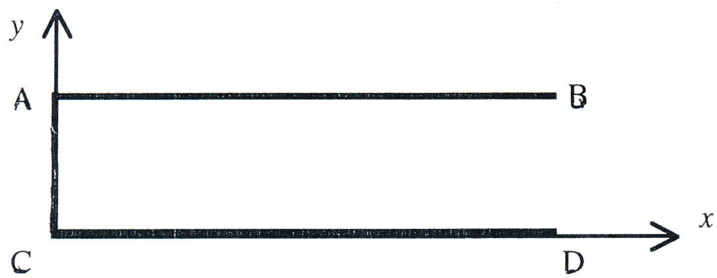
Last term can be dropped because it does not contribute ~~to~~ V.

Problem 2 (10 marks)

Consider the arrangement shown in the figure below, where AB and CD are parallel boundary plates from $x = 0$ to $x = \infty$ along which the potential V is zero everywhere. Along the boundary line AC (of length L) at $x = 0$ the fixed potential is $V = V_0 \sin(3\pi y/L)$. Assuming no variation in the z direction, you are given that

$$V(x,y) = [ae^{\lambda x} + be^{-\lambda x}][c\sin(\lambda y) + d\cos(\lambda y)] \quad \text{for } x > 0 \text{ and } 0 < y < L \ (a, b, c, d \text{ constants})$$

represents the general solution from the method of separation of variables (where λ is a positive constant). By now applying the boundary conditions along AB, CD, and AC, as well as the extra boundary condition (which you should state) as $x \to \infty$, find the specific solution for $V(x,y)$.



$V = 0$ everywhere at $y = 0$ $\Rightarrow$ $d = 0$

As $x \to \infty$ (any y) we must have extra b.c. that $\underline{V \to 0 \ (\text{as } x \to \infty)}$

$\therefore$ This implies $a = 0$ (taking $\lambda > 0$)

$\therefore$ The solution so far simplifies to $V(x,y) = bc \sin(\lambda y) e^{-\lambda x}$

From the b.c. at $x = 0$ $\Rightarrow$ $bc \sin(\lambda y) = V_0 \sin(3\pi y/L)$

$\therefore$ $bc = V_0$ and $\lambda = 3\pi/L$

As a check, along AB we have $y = L$ so $\sin\left(\frac{3\pi y}{L}\right) \to \sin(3\pi) = 0$ as required

$\therefore$ Solution is $\underline{V(x,y) = V_0 e^{-(3\pi x/L)} \sin(3\pi y/L)}$

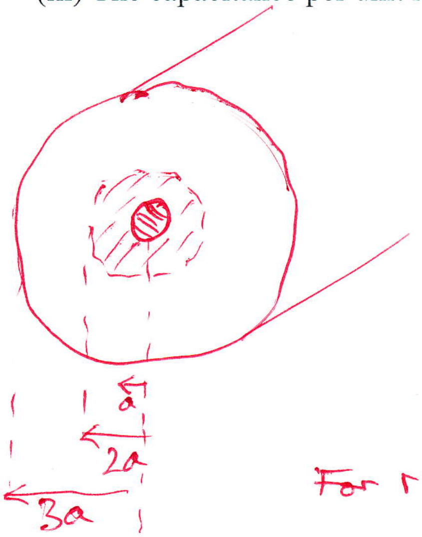
Problem 3 (10 marks)

A long cylindrical capacitor is formed as follows. A solid conducting cylinder of radius a is coated with a linear dielectric layer of thickness a. Next there is a vacuum layer of thickness a, which is is surrounded coaxially by a thin conducting shell of radius $3a$. Given that the inner conductor has charge $+Q$ and the outer conductor charge $-Q$ and that the dielectric has relative permittivity ε_r, calculate

(i) The electric displacement **D** in the 4 regions ($r < a$, $a < r < 2a$, $2a < r < 3a$, $r > 3a$).

(ii) The electric field **E** in each region.

(iii) The capacitance per unit length.



(i) Use Gauss's law in form $\int \underline{D} \cdot d\underline{a} = Q_{f\text{-}encl}$

$\Rightarrow 2\pi r D = Q_{f\text{-}enc}$ for any r

For $r < a$, $Q_{f\text{-}enc} = 0 \Rightarrow \underline{D = 0}$

For $a < r < 2a$, $Q_{fencl} = Q$ (per unit length)

$\therefore \underline{D = \dfrac{Q}{2\pi r}}$ (radially out)

For $2a < r < 3a$, $Q_{f\text{-}encl} = Q$ $\therefore \underline{D = \dfrac{Q}{2\pi r}}$ (radially out)

For $r > 3a$, $Q_{f\text{-}enc} = Q - Q = 0$

$\therefore \underline{D = 0}$

(ii) Use $\underline{D} = \varepsilon_0 \varepsilon_r \underline{E}$ in the linear dielectric, and $\underline{D} = \varepsilon_0 \underline{E}$ in vacuum

$\therefore D = 0 \Rightarrow \underline{E = 0}$ in regions $r < a$ and $r > 3a$

For $a < r < 2a$, $\underline{E = \dfrac{Q}{2\pi \varepsilon_0 \varepsilon_r r}}$ (radially out)

For $2a < r < 3a$, $\underline{E = \dfrac{Q}{2\pi \varepsilon_0 r}}$ (radially out

(iii) We need difference in V between the conductors, then $C = Q/V$

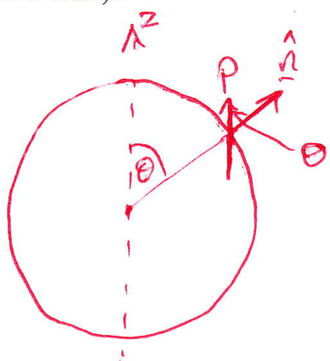
Integrating along a radial line, $V = -\int E\, dr$

$$= \frac{Q}{2\pi\varepsilon_0\varepsilon_r}\int_a^{2a}\frac{dr}{r} + \frac{Q}{2\pi\varepsilon_0}\int_{2a}^{3a}\frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0\varepsilon_r}\left\{\left[\ln r\right]_a^{2a} + \varepsilon_r\left[\ln r\right]_{2a}^{3a}\right\}$$

$$= \frac{Q}{2\pi\varepsilon_0\varepsilon_r}\left\{\ln 2 + \varepsilon_r \ln(3/2)\right\}$$

$$\therefore \underline{C = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln(2) + \varepsilon_r \ln(3/2)}}$$

Short question 1 (3 marks)
A sphere (with radius R) of dielectric material has a "frozen-in" polarization that has a constant magnitude P_0 and a fixed direction (along the z axis). In this case what is the volume bound charge density inside the sphere? Also, what is the surface bound charge density (take a general point on the surface that makes polar angle θ with the z axis)?



$\rho_b = -\nabla \cdot \underline{P} = \underline{0}$ since $\underline{P}$ is a constant

$\sigma_b = \underline{P} \cdot \underline{\hat{n}} = \underline{P_0 \cos\theta}$

Short question 2 (3 marks)
State briefly what the two uniqueness theorems in electrostatics tell us. Why are they useful in problem solving (e.g., using techniques such as the image method or the separation of variables method)?

The 2 theorems state that the solutions of Laplace's equation (with boundary conditions specified) lead to <u>unique</u> results for potential and electric field (even in the presence of conductors, for the 2nd theorem).
So, if you find a solution by whatever method, then it is <u>the</u> solution.

Short question 3 (4 marks)
A hollow conductor is electrically neutral. A point charge Q is placed inside the cavity of the conductor. Indicate true (T) or false (F) for each of the following:

F (a) There is no electric field in the cavity.
F (b) There is no electric field outside the cavity.
F (c) The electric field outside does not depend on the magnitude or position of the charge in the cavity.
F (d) If the conductor is now grounded (earthed), none of the above responses will be different.

Electromagnetic Theory I (Physics 3300A) 2012-2013

Formula Sheet (for Midterm 2)

(You will be provided with a copy of this Formula Sheet for the test. Do NOT bring your own copy)

Section2

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r_e^2}\hat{r}_e \qquad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{volume}}\frac{\rho(\mathbf{r}')}{r_e^2}\hat{r}_e d\tau' \qquad \oint_{\text{surface}}\mathbf{E}\cdot da = \frac{1}{\varepsilon_0}Q_{\text{enc}}$$

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho \qquad \mathbf{E} = -\nabla V \qquad \nabla^2 V = -\frac{\rho}{\varepsilon_0} \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{r_e}d\tau'$$

$$\mathbf{E}_{above} - \mathbf{E}_{below} = \frac{1}{\varepsilon_0}\sigma\hat{\mathbf{n}} \qquad W = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1(i\neq j)}^{n}\frac{q_i q_j}{r_{eij}} \qquad W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i) \qquad \nabla\times\mathbf{E} = 0$$

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2\, d\tau \qquad \mathbf{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{\mathbf{n}} \qquad P = \frac{\varepsilon_0}{2}E^2 \qquad C = \frac{Q}{V} \qquad W = \frac{1}{2}CV^2$$

$$W = \frac{1}{2}\int \rho V\, d\tau$$

Section 3

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int\rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\,\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots\right]$$

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \qquad Q = \int\rho(\mathbf{r}')d\tau' \qquad \mathbf{p} = \int\mathbf{r}'\rho(\mathbf{r}')d\tau'$$

$$\mathbf{p} = \sum_{i=1}^{n} q_i\mathbf{r}_i' \qquad V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \qquad \mathbf{p} = q\mathbf{d}$$

Section 4

$$\mathbf{p} = \alpha\mathbf{E} \qquad \mathbf{N} = \mathbf{p}\times\mathbf{E} \qquad \mathbf{F} = \mathbf{p}\cdot\nabla\mathbf{E} \qquad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \qquad \rho_b = -\nabla\cdot\mathbf{P} \qquad \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$$

$$\nabla\cdot\mathbf{D} = \rho_f \qquad \oint\mathbf{D}\cdot da = Q_{f\text{-enc}} \qquad \mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \qquad \mathbf{D} = \varepsilon\mathbf{E} \qquad \varepsilon = \varepsilon_r\varepsilon_0$$

$$\varepsilon_r = 1 + \chi_e \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V\frac{\hat{\mathbf{r}}_c\cdot\mathbf{P}(\mathbf{r}')}{r_c^2}d\tau'$$