Calculus 1100a Midterm Exam and Solutions

A1 [**2 marks**] Evaluate $\ln e^{3\ln 2}$

A: 9	B: 8	#C: $3\ln 2$	D: e^3+2	E: 6

<u>Answer</u> $\ln e^{3\ln 2} = \ln\left[(e^{\ln 2})^3\right] = 3\ln(e^{\ln 2}) = 3\ln 2$

A2 [**2 marks**] Evaluate $\log_3 12 + \log_3 6 - 3\log_3 2$

A: 8	B: 12	C: 9	D: 0	#E: 2

<u>Answer</u> $3\log_3 2 = \log_3(2^3) = \log_3(8)$

$$\Rightarrow \log_3 12 + \log_3 6 - 3\log_3 2 = \log_3 12 + \log_3 6 - \log_3 8 = \log_3\left(\tfrac{(12)(6)}{8}\right)$$
$$= \log_3(9) = \log_3(3^2) = 2$$

A3 [**2 marks**] Find the exact value of $\arcsin\left(-\tfrac{1}{2}\right)$

A: $-\frac{1}{2}$	B: $-\frac{\sqrt{3}}{2}$	C: 1	#D: $-\frac{\pi}{6}$	E: $-\frac{\pi}{3}$

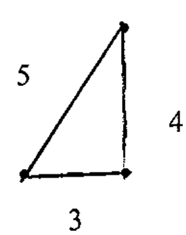
<u>Answer</u> $\sin^{-1}\left(-\tfrac{1}{2}\right) = y \Rightarrow \sin y = -\tfrac{1}{2}$ where $-\tfrac{\pi}{2} \le y \le 0$

$$\Rightarrow y = -\tfrac{\pi}{6}$$

A4 [**2 marks**] Find the exact value of $\cos[\sin^{-1}(\tfrac{4}{5})]$

#A: $\frac{3}{5}$	B: $\frac{4}{5}$	C: $-\frac{3}{5}$	D: $-\frac{4}{5}$	E: 1

<u>Answer</u> Consider the triangle



Then

$$\sin(\theta) = \tfrac{4}{5} \Rightarrow \sin^{-1}(\tfrac{4}{5}) = \theta$$
$$\Rightarrow \cos[\sin^{-1}(\tfrac{4}{5})] = \cos\theta = \tfrac{3}{5}$$

A5 [**2 marks**] Find the exact value of $\tan\left[\tan^{-1}\left(\tfrac{3}{4}\right)\right]$

#A: $\frac{3}{4}$	B: $\frac{1}{4}$	C: $\frac{5}{4}$	D: $\frac{7}{4}$	E: $-\frac{1}{4}$

Answer $\tan(\tan^{-1}x) = x$ for all x (cancellation law) $\Rightarrow \tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right] = \frac{3}{4}$

A6 $\left[\right.$**2 marks**$\left.\right]$ Find the exact value of $\tan^{-1}\left[\tan\left(\frac{3\pi}{4}\right)\right]$

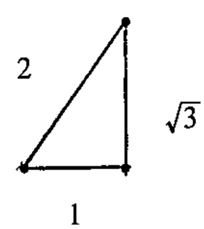
A: $\frac{3\pi}{4}$	B: $\frac{\pi}{4}$	#C: $-\frac{\pi}{4}$	D: $-\frac{3\pi}{4}$	E: 1

Answer: Given any x then to determine $\tan^{-1}(x)$ we need to find $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ so that $\tan(\theta) = x$. Thus to determine $\tan^{-1}\left[\tan\left(\frac{3\pi}{4}\right)\right]$ we want to find $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ so that $\tan(\theta) = \tan\left(\frac{3\pi}{4}\right)$ Since tan is periodic of period π it follows that $\tan\left(\frac{3\pi}{4}\right) = \tan\left(-\frac{\pi}{4}\right)$ and, hence $\theta = -\frac{\pi}{4}$ is the answer.

A7 $\left[\right.$**2 marks**$\left.\right]$ Find the exact value of $\cos^{-1}\left[\sin\left(\frac{\pi}{6}\right)\right]$

A: $\frac{\pi}{6}$	B: $-\frac{\pi}{6}$	#C: $\frac{\pi}{3}$	D: $-\frac{\pi}{3}$	E: $\frac{\pi}{2}$

Answer Consider the triangle



We have $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$. Thus $\cos^{-1}\sin\left(\frac{\pi}{6}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

A8 $\left[\right.$**2 marks**$\left.\right]$ Find a formula for the inverse $f^{-1}(x)$ of the function $f(x) = e^{2x+1}$.

A $\frac{2}{1-\ln x}$	#B $\frac{(\ln x)-1}{2}$	C $\frac{2}{1+\ln x}$	D $\frac{(\ln x)+1}{2}$	E $\frac{1-\ln x}{2}$

Answer: $y = e^{2x+1} \Rightarrow \ln y = 2x + 1$

$$\Rightarrow x = \frac{(\ln y) - 1}{2}$$

$$\Rightarrow y = \frac{(\ln x) - 1}{2}$$

A9 [**2 marks**] $\tan\left[2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] =$ __________________?

#A $\sqrt{3}$	B $-\sqrt{3}$	C $\frac{1}{\sqrt{3}}$	D $\frac{1}{2}$	E $-\frac{1}{\sqrt{3}}$

Answer: $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \Rightarrow 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$

$$\Rightarrow \tan\left[2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Consider the four graphs below. For each of the following four functions choose the letter which labels its graph. Be sure to transfer each of your answers to the scantron sheet.

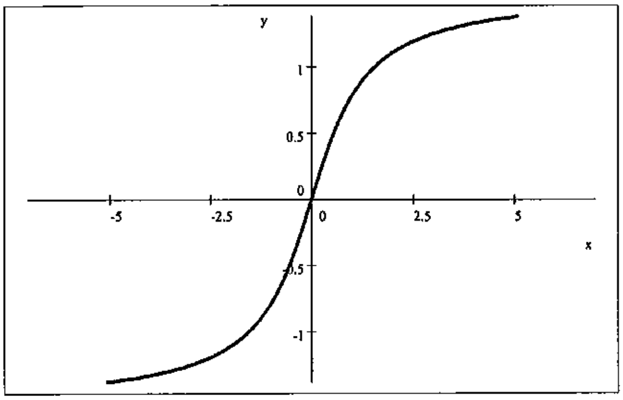
A10 [**2 marks**] The graph of $y = e^{-x}$ is ___D___

A11 [**2 marks**] The graph of $y = e^{-x^2}$ is ___B___

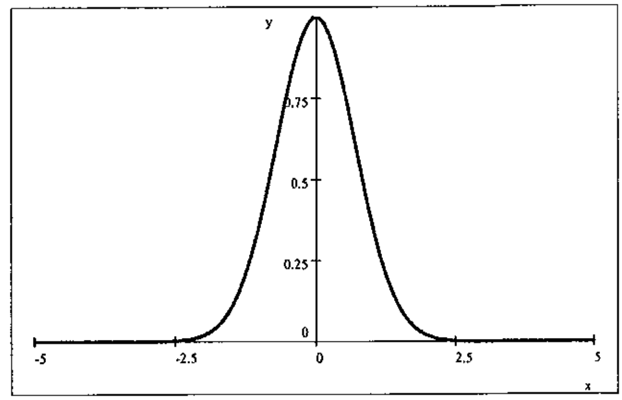
A12 [**2 marks**] The graph of $y = \tan^{-1}(x)$ is ___A___

A13 [**2 marks**] The graph of $y = e^{|x|}$ is ___C___

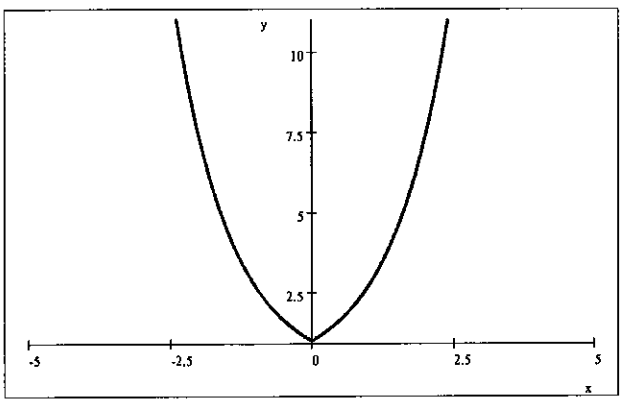
A



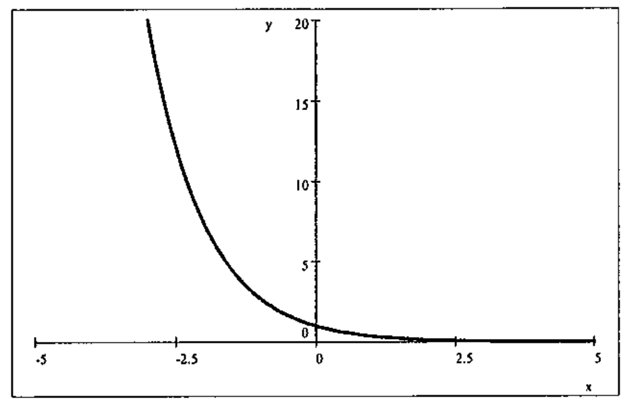
B



C



D



A14 [**2 marks**] Determine $\lim_{x \to 5} \dfrac{3}{|x-5|}$

| #A ∞ | B $-\infty$ | C $-\frac{1}{2}$ | D $\frac{1}{2}$ | E 0 |

<u>Answer</u>: $\lim_{x \to 5} |x-5| = 0 \Rightarrow \lim_{x \to 5} \dfrac{3}{|x-5|} == \infty$

A15 [**2 marks**] Determine $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

| A ∞ | B: $-\infty$ | C 0 | #D: 4 | E -4 |

Answer: $$\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} (x+2) = 4$$

A16 [**2 marks**] Determine $\lim_{x \to \infty} \sin(\tan^{-1}x)$

#A: 1	B -1	C $\frac{1}{\sqrt{2}}$	D ∞	E: $-\infty$

Answer: $$\lim_{x \to \infty} \tan^{-1}x = \tfrac{\pi}{2} \Rightarrow \lim_{x \to \infty} \sin(\tan^{-1}x) = \sin\left(\tfrac{\pi}{2}\right) = 1$$

A17 [**2 marks**] Determine $\lim_{x \to 0^+} \ln(\tan x)$

A 0	B 1	C -1	D ∞	#E: $-\infty$

Answer: $$\lim_{x \to 0^+} \tan x = 0 \Rightarrow \lim_{x \to 0^+} \ln(\tan x) = -\infty$$

A18 [**2 marks**] Determine $\lim_{x \to \infty} \dfrac{3x^8+7x^6+3x^3+1}{9x^8+5x^4+2x}$.

A: ∞	B: $-\infty$	C: 0	#D: $\frac{1}{3}$	E: $\frac{1}{2}$

Answer:

$$\lim_{x \to \infty} \frac{(3x^8+7x^6+3x^3+1)\left(\dfrac{1}{x^8}\right)}{(9x^8+5x^4+2x)\left(\dfrac{1}{x^8}\right)} = \lim_{x \to \infty} \frac{(3+7/x+3/x^5+1/x^8)}{(9+5/x^4+2/x^7)} = \frac{3}{9} = \frac{1}{3}$$

A19 [**2 marks**] Determine $\lim\limits_{x \to \infty} \dfrac{\sqrt{3x^2 - 5}}{2x - 3}$

A: ∞ | B: $-\infty$ | C: 0 | D: $\frac{5}{3}$ | #E: $\frac{\sqrt{3}}{2}$

Answer:

$$\lim_{x \to \infty} \frac{\sqrt{3x^2 - 5}}{2x - 3} = \lim_{x \to \infty} \frac{\left(\sqrt{3x^2 - 5}\right)\left(\dfrac{1}{x}\right)}{(2x - 3)\left(\dfrac{1}{x}\right)} = \lim_{x \to \infty} \frac{\sqrt{3 - 1/x^2}}{2 - 3/x} = \frac{\sqrt{3}}{2}$$

A20 [**2 marks**] Determine $\lim\limits_{x \to \infty} \sqrt{\dfrac{x + 8x^2}{2x^2 - 1}}$

A: ∞ | B: $-\infty$ | #C: 2 | D: $\sqrt{8}$ | E: $\frac{1}{2}$

Answer: $$\lim_{x \to \infty} \sqrt{\frac{x + 8x^2}{2x^2 - 1}} = \lim_{x \to \infty} \sqrt{\frac{(x + 8x^2)\left(\dfrac{1}{x^2}\right)}{(2x^2 - 1)\left(\dfrac{1}{x^2}\right)}} = \lim_{x \to \infty} \sqrt{\frac{(1/x) + 8}{2 - 1/x^2}} = \sqrt{\frac{8}{2}} = =$$

A21 [**2 marks**] Determine $\lim\limits_{x \to -\infty} 3^{1/x}$

A: 0 | #B: 1 | C: -1 | D: 2 | E: Does Not Exist

Answer: $$\lim_{x \to -\infty} 3^{1/x} = \lim_{t \to 0^-} 3^{t} = 1$$

A22 [**2 marks**] Determine $\lim\limits_{x \to (-4)^+} \dfrac{|x + 4|}{x + 4}$

#A: 1 | B: -1 | C: ∞ | D: $-\infty$ | E: Does Not Exist

Answer: $\lim_{x \to (-4)^+} \frac{|x+4|}{x+4} = \lim_{x \to (-4)^+} \frac{x+4}{x+4} = \lim_{x \to (-4)^+} 1 = 1$

A23 [**2 marks**] Determine $\lim_{x \to \infty} \sin(1/x)$.

#A: 0 | B: ∞ | C: $-\infty$ | D: 1 | E: Does Not Exist

Answer: $\lim_{x \to \infty} \sin(1/x) = \lim_{t \to 0^+} \sin(t) = 0$

A24 [**2 marks**] Determine $\lim_{x \to -\infty} \tan^{-1}(-x^2)$.

A: $\frac{\pi}{2}$ | B: ∞ | C: $-\infty$ | #D: $-\frac{\pi}{2}$ | E: Does Not Exist

Answer: let $t = -x^2$

$x \to -\infty \Rightarrow x^2 \to \infty \Rightarrow t = -x^2 \to -\infty$

Hence $\lim_{x \to -\infty} \tan^{-1}(-x^2) = \lim_{t \to -\infty} \tan^{-1}(t) = -\pi/2$

A25 [**2 marks**] If $f(x) = \ln(\arcsin x)$, find $f'(x)$

A: $\frac{1}{\arcsin x}$ | B: $\frac{-1}{\arcsin x}$ | C: $\frac{1}{(1+x^2)\arcsin x}$ | #D: $\frac{1}{\left(\sqrt{1-x^2}\right)\arcsin x}$ | E: $\frac{-1}{\left(\sqrt{1-x^2}\right)\text{ar}}$

Answer: $f(x) = \ln(\sin^{-1}x) = \ln(y)$ where $y = \sin^{-1}x$

$\Rightarrow f'(x) = \frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin^{-1}x}\frac{1}{\sqrt{1-x^2}}$

A26 [**2 marks**] Find $\frac{d}{dx}[\tan^{-1}(x^2)]$

#A $\dfrac{2x}{1+x^4}$	B $2x\cos^{-1}(x^2)$	C $\cos^{-1}(x^2)$	D $\dfrac{2x}{\sqrt{1-x^4}}$	E $-\dfrac{1}{1+x^4}$

Answer: $f(x) = \tan^{-1}(x^2) = \tan^{-1}(y)$ where $y = x^2$

$$\Rightarrow f'(x) = \frac{1}{1+y^2}\frac{dy}{dx} = \frac{1}{1+x^4}(2x)$$

A27 [**2 marks**] Find $\dfrac{d}{dx}(\sin^{-1}x)^2$

A: $2(\sin^{-1}x)\csc x\cot x$	B: $-2(\sin^{-1}x)^3$	C: $2(\sin^{-1}x)$	#D: $\dfrac{2(\sin^{-1}x)}{\sqrt{1-x^2}}$	E: $2(\sin^{-1}x)(\cos^{-1}x)$

Answer: $f(x) = (\sin^{-1}x)^2 = y^2$ where $y = \sin^{-1}x$

$$\Rightarrow f'(x) = 2y\frac{dy}{dx} = 2\sin^{-1}x\frac{1}{\sqrt{1-x^2}}$$

A28 [**2 marks**] Suppose $F(x) = f(g(x))$ and $f(2) = 0$, $f'(2) = 3$, $f'(7) = 6$, $g(2) = 7$, $g'(2) = 4$ and $g'(7) = 9$. Find $F'(2)$.

A: 0	B 18	C: 21	D: 42	#E: 24

Answer: $F(x) = f(g(x)) \Rightarrow F'(a) = f'(g(a))g'(a)$

$$\Rightarrow F'(2) = f'(g(2))g'(2) = \left[f'(7)\right](4) = (6)(4) = 24$$

A29 [**2 marks**] Suppose $f(x) = 3^{\tan^{-1}(x)}$. Find $f'(x)$

A $3^{\tan^{-1}(x)}$	B: $\dfrac{3^{\tan^{-1}(x)}}{1+x^2}$	C $\ln(3)3^{\tan^{-1}(x)}$	#D $\dfrac{\ln(3)3^{\tan^{-1}(x)}}{1+x^2}$	E $3^{\tan^{-1}(x)}$ s

Answer: $f(x) = 3^{\tan^{-1}(x)} = 3^y$ where $y = \tan^{-1}(x)$

$$\Rightarrow f'(x) = (\ln 3)3^y \frac{dy}{dx} = (\ln 3)3^{\tan^{-1}(x)} \left(\frac{1}{1+x^2}\right)$$

A30 [**2 marks**] Determine $\lim_{x \to 0} \frac{\sin(3x)}{4x}$

A 1	#B: $\frac{3}{4}$	C: 3	D: $\frac{1}{4}$	E: Does not Exist

Answer: Can use L'Hospital's Rule

$$\lim_{x \to 0} \frac{\sin(3x)}{4x} \overset{(H)}{=} \lim_{x \to 0} \frac{3\cos(3x)}{4} = \frac{3(1)}{4} = \frac{3}{4}$$

A31 [**2 marks**] Determine $\lim_{x \to 0} \frac{\tan(3x)}{\sin(2x)}$

#A $\frac{3}{2}$	B: $\frac{1}{2}$	C: 3	D: 1	E: Does not Exist

Answer: Can use L'Hospital's Rule

$$\lim_{x \to 0} \frac{\tan(3x)}{\sin(2x)} \overset{(H)}{=} \lim_{x \to 0} \frac{3\sec^2(3x)}{2\cos(2x)} = \frac{3(1)^2}{2(1)} = \frac{3}{2}$$

A32 [**2 marks**] Determine $\frac{d}{dx}[\sin^{-1}(\cos(x))]$.

A 1	B −2	C 0	#D −1	E $\frac{1}{\sqrt{1-\cos^2 x}}$

Answer: $f(x) = \sin^{-1}(\cos(x)) = \sin^{-1}(y)$ where $y = \cos(x)$

$$\Rightarrow f'(x) = \left(\frac{1}{\sqrt{1-y^2}}\right)\frac{dy}{dx} = \left(\frac{1}{\sqrt{1-\cos^2 x}}\right)(-\sin x)$$

$$= \left(\frac{1}{\sqrt{\sin^2 x}}\right)(-\sin x) = \frac{-\sin x}{\sin x} = -1$$

A33 $\left[\textbf{2 marks}\right]$ If $y = 2^{\ln x}$ evaluate $\dfrac{dy}{dx}$ at $x = e$

#A $\frac{2\ln 2}{e}$	B: $\frac{e\ln 2}{2}$	C: $\frac{e}{2\ln 2}$	D: $\frac{\ln 2}{2e}$	E: 2

Answer: $f(x) = 2^{\ln x} = 2^y$ where $y = \ln x$

$$\Rightarrow f'(x) = (\ln 2)2^y \frac{dy}{dx} = (\ln 2)2^{\ln x}\frac{1}{x}$$

$$\Rightarrow f'(e) = (\ln 2)2^{\ln e}\frac{1}{e} = \frac{2\ln 2}{e}$$ (observe $\ln e = 1$)

A34 $\left[\textbf{2 marks}\right]$ Let $f(x) = \ln(e^x + 1)$. Find $f'(0)$

A $\ln 2$	#B $\frac{1}{2}$	C 1	D $-\frac{1}{2}$	E -1

Answer: $f(x) = \ln(e^x + 1) = \ln y$ where $y = e^x + 1$

$$\Rightarrow f'(x) = \frac{1}{y}\frac{dy}{dx} = \frac{1}{e^x + 1}e^x$$

$$\Rightarrow f'(0) = \frac{e^0}{e^0 + 1} = \frac{1}{2}$$

A35 $\left[\textbf{2 marks}\right]$ Which of the following is an antiderivative of $f(x) = 5x(2 - \sqrt{x})$

A $10x^{3/2} - x^{5/2}$	B $10x^2 - 2x^{5/2}$	#C $5x^2 - 2x^{5/2}$	D $5x^2 - \frac{25}{2}x^{5/2}$	E $5x - 2x^{3/2}$

Answer: We have $f(x) = 5x(2 - \sqrt{x}) = 10x - 5x^{3/2}$. Hence the antiderivatives of $f(x)$ are

$$F(x) = 10\left(\frac{x^2}{2}\right) - 5\frac{x^{5/2}}{5/2} + C = 5x^2 - \frac{2}{5}(5x^{5/2}) + C = 5x^2 - 2x^{5/2} + C$$

B1 $\left[\textbf{6 marks}\right]$ Consider the function given by $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{for } x < 0 \\ 3x^2 + 1 & \text{for } 0 \le x \le 2 \\ x + 5 & \text{for } x > 2 \end{cases}$

State the value of the indicated limit, if it exists, in the space provided. If a limit does not exist write *DNE*.

(a) $\lim_{x \to 0^+} f(x) =$ ___1___

(b) $\lim_{x \to 0^-} f(x) =$ ___1___

(c) $\lim_{x \to 0} f(x) =$ ___1___

(d) $\lim_{x \to 2^+} f(x) =$ ___7___

(e) $\lim_{x \to 2^-} f(x) =$ ___13___

(f) $\lim_{x \to 2} f(x) =$ ___*DNE*___

B2 [**4 marks**] The limit $\lim_{h \to 0} \dfrac{(8+h)^{1/3} - 2}{h}$ is the derivative of some function f at some number a. Then

$f(x) =$ __________?

$a =$ __________?

Answer: Since the definition of the derivative can be stated as

$f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ it follows that $\lim_{h \to 0} \dfrac{(8+h)^{1/3} - 2}{h}$ is the derivative for the case

$f(x) = \underline{x^{1/3}}$?

$a = \underline{8}$

B3 [**5 marks**] Find $\dfrac{dy}{dx}$ if $y = \tan^{-1}\left(e^{\cos^2 x}\right)$ DO NOT SIMPLIFY YOUR ANSWER..

<u>Answer</u>: $$\frac{dy}{dx} = \left[\frac{1}{1+\left(e^{\cos^2 x}\right)^2}\right]\left[e^{\cos^2 x}\right][2\cos x][-\sin x]$$

B4 [**5 marks**] Find $\dfrac{dy}{dx}$ as a function of x if $y = (\sqrt{x})^{\cos(2x)}$.

<u>Answer</u>: Applying ln to both sides of $y = (\sqrt{x})^{\cos(x)}$ one obtains

$$\ln y = \ln\left((\sqrt{x})^{\cos(2x)}\right) = \cos(2x)\ln(\sqrt{x}) = \cos(2x)\ln\left(x^{1/2}\right) = \frac{\cos(2x)\ln(x)}{2}$$

Differentiating the equation $y = \dfrac{\cos(2x)\ln(x)}{2}$, one obtains

$$\frac{1}{y}y' = \frac{1}{2}\left[-2\sin(2x)\ln(x) + \cos(2x)\left(\frac{1}{x}\right)\right] = \frac{\cos(2x)}{2x} - \sin(2x)\ln(x)$$

Cross multiplying and substituting $y = (\sqrt{x})^{\cos(2x)}$, one obtains

$$y' = \left((\sqrt{x})^{\cos(2x)}\right)\left(\frac{\cos(2x)}{2x} - \sin(2x)\ln(x)\right)$$

B5 [**6 marks**] (a) Suppose the function $y = f(x)$ satisfies the equation $\cos(x - y) = xy$.

Determine $\dfrac{dy}{dx}$ as an expression in x and y.

(b) Find the equation of the tangent line to the curve $\cos(x-y) = xy$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Answer: (a) Applying implicit differentiation to the above equation one has:

$$[-\sin(x-y)](1-y') = y + xy'$$

This can be rewritten as

$$\sin(x-y)y' - xy' = y + \sin(x-y)$$

and hence as

$$(\sin(x-y) - x)y' = y + \sin(x-y)$$

Thus we have

$$y' = \frac{y + \sin(x-y)}{\sin(x-y) - x}$$

(b) If $x = \frac{\pi}{2}$ and $y = 0$ one has $y' = \dfrac{0 + \sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right) - \frac{\pi}{2}} = \dfrac{1}{1 - \frac{\pi}{2}}$. Hence the equation for the tangent line through $\left(\frac{\pi}{2}, 0\right)$.is given by $y - 0 = \left(\dfrac{1}{1-\frac{\pi}{2}}\right)\left(x - \frac{\pi}{2}\right)$ i.e.

$$y = \left(\frac{1}{1-\frac{\pi}{2}}\right)\left(x - \frac{\pi}{2}\right)$$

B6 [5 marks] Find all values of x in the interval $[0, \pi]$ that satisfy the equation $\sec^2(x) = 4$.

Answer: $\sec^2(x) = 4 \Rightarrow \cos^2(x) = 1/4$

$\Rightarrow \cos(x) = \pm 1/2$

$\Rightarrow$ for the range $0 < x < \pi$ one must have $x = \pi/3$ or $x = 2\pi/3$