Model of the Earth with a Constant-Density Core and a Constant-Gravity Mantle

Carol Evelyn Millar[1,2] and Romas Mitalas[1,3]

ABSTRACT

A model of the earth, consisting of a constant density core and a mantle in which the magnitude of gravity is constant, is developed and compared with the earth. The assumption of constant magnitude for gravity determines the density distribution in the mantle and all the physical properties of the mantle. The total mass of the earth then gives the density of the core in terms of the fractional radius of the core. The resulting pressure distribution as well as the moments of inertia of the core and mantle of the model are compared with those of the earth. The pressure and density distributions give a constant bulk modulus in the mantle. Assumption of the Poisson's ratio gives the speeds of seismic waves in the mantle, which decrease as $r^{1/2}$ in the mantle. Suggestions to use this model as a pedagogical tool in geophysical education are made.

Key words: Education - undergraduate; Geophysics - solid earth; Problems - geomathematical.

1. Introduction

Models of the earth have a long history beginning in the last century with Wiechert (1897), who proposed a model consisting of a constant density, rocky mantle superimposed on a constant density, iron core. The relative proportions of mass were based on the relative abundance of iron in meteorites. Even though the abrupt decrease of seismic velocities deep in the earth allowed Gutenberg (1914) to determine the depth of the core-mantle boundary, the location of the boundary was not incorporated in models for at least a

[1]University of Western Ontario, Department of Physics and Astronomy, London, Ontario, Canada N6A 3K7

[2]cmillar@io.astro.uwo.ca

[3]rmitalas@.astro.uwo.ca

decade. Jeffreys (1924) fitted the mass and moment of inertia of the earth to the same two-constant density model of the earth and obtained $r = 0.78R_\oplus$, where $R_\oplus$ is the radius of the earth, for the location of the density discontinuity; the resulting density of the core was $\rho_c = 8.22 \times 10^3\,\mathrm{kg\,m^{-3}}$ for the adopted density of the mantle $\rho_m = 3.12 \times 10^3\,\mathrm{kg\,m^{-3}}$. Five years later using the same model, Jeffreys (1929) set the core-mantle boundary at $r = 0.545R_\oplus$ and fitted again the mass and moment of inertia of the earth to obtain $\rho_c = 12.04 \times 10^3\,\mathrm{kg\,m^{-3}}$ and $\rho_m = 4.27 \times 10^3\,\mathrm{kg\,m^{-3}}$ for the densities of the core and the mantle. The two-constant density model is still in use (Schubert and Spohn, 1990).

In the meantime, it was shown by Williamson and Adams (1923) that the distribution of seismic velocities allowed the density distribution of the earth to be determined. Improvements in the knowledge of the density distribution based on accumulating seismological data continued (Bullen, 1975). The observation of free oscillations of the earth beginning in 1960 with the great Chilean earthquake provided new observational data to constrain the models of the earth. The result was the parametric earth model (Dziewonski, Hales and Lapwood, PEM 1975) which we use for comparison. It is presented as Appendix G of a textbook by Stacey (1976). A recent, critical discussion of the density determination in the earth was given by Bolt (1991). A historical account of discovery and development of ideas about the structure, physical state and models of the earth was presented by Brush (1980).

We consider a model of the earth consisting of a constant density core surrounded by a mantle in which the magnitude of gravity is constant. The approximate constancy of magnitude of gravity in the mantle had been noticed a long time ago by Benfield (Jeffreys, 1929). The consequences, to the best of our knowledge, have not been worked out. The physical basis of the assumption of constant magnitude of local gravity in the mantle can be seen in Fig. 1, where local gravity, based on the PEM model of the earth, is plotted. The core-mantle boundary is indicated. The maximum deviation of gravity in the mantle from the surface value is 8%. Since gravity decreases linearly in a constant density sphere, the almost linear decrease of gravity in the core justifies the assumption of constant density there. We work out the physical consequences of this model and compare them with the earth. The model is a useful pedagogical tool to illustrate the principles of hydrostatic equilibrium and acquaint the students with the gross features of the earth. The analytical results of the model can be derived in class or assigned as homework. Other problems suggest themselves. For example, how well does the two-constant density model of Jeffreys (1929) reproduce the gravity in the mantle (Snyder, 1985)? Hereafter, we will use constant gravity to denote that the magnitude of gravity is constant.

2. Constant gravity mantle

The assumption of constant gravity in a spherically symmetric mass distribution determines the density distribution from the equation of mass conservation,

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \quad , \tag{1}$$

where $M(r)$ is the mass interior to radius r and $\rho(r)$ the density; M and R will denote the total mass and radius.

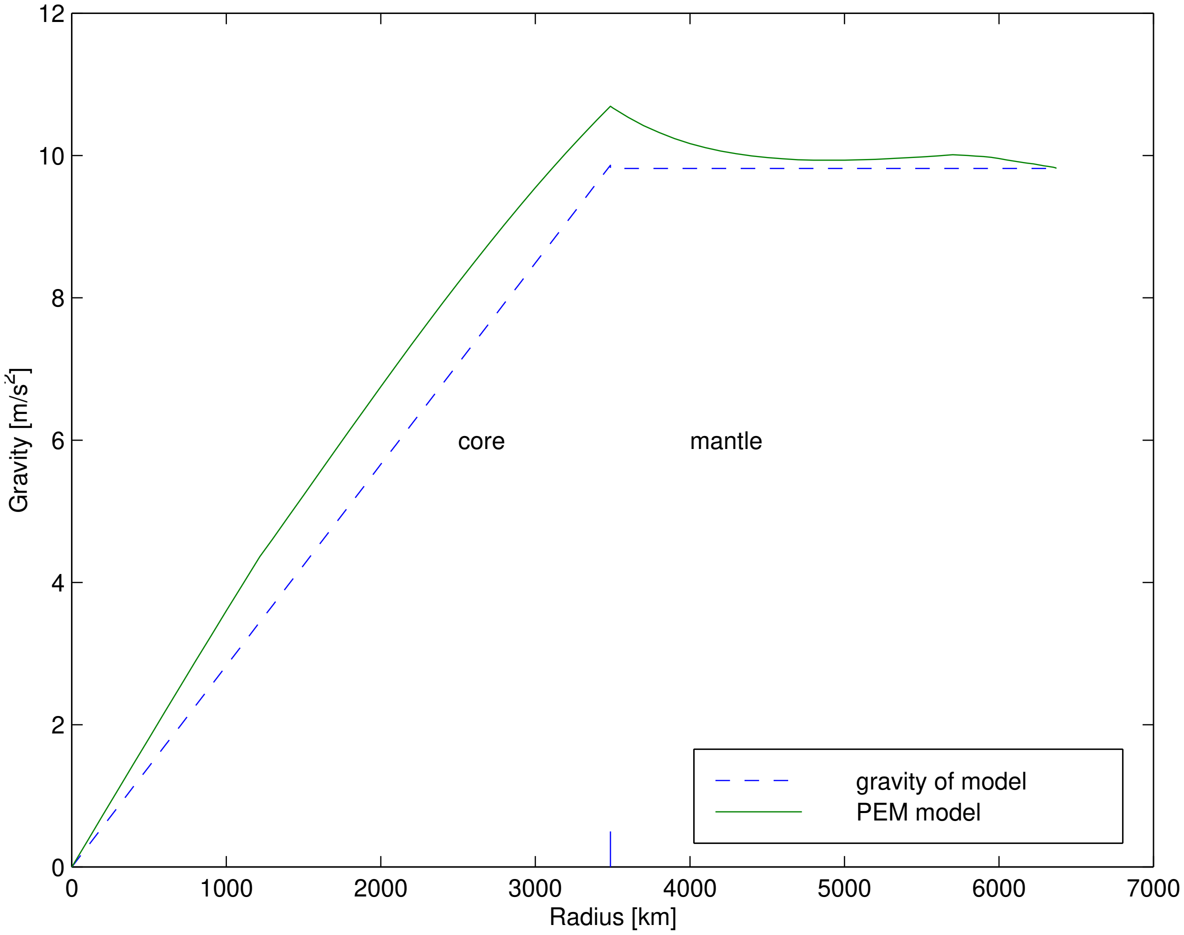

Fig. 1.— The local gravity of the parametric earth model, (PEM), drawn as a solid line, is compared with the assumed constant gravity of the mantle and the gravity of the constant density core, drawn as a dashed line.

The assumption of constant gravity,

$$g(r) = \frac{GM(r)}{r^2} = \frac{GM}{R^2} \equiv g_0 \quad , \tag{2}$$

determines the mass distribution, $M(r) = (M/R^2)r^2$. Then, from the equation of mass conservation, the density distribution in a region where gravity is constant is

$$\rho(r) = \frac{2}{3}\left(\frac{R}{r}\right)\bar{\rho} \quad , \tag{3}$$

where $\bar{\rho}$ is the mean density of M. Thus the density is an inverse function of the radius. All the physical properties of the mantle now follow.

The surface density and the mean density of the mass distribution are simply related, $\rho_s \equiv \rho(R) = (2/3)\bar{\rho}$, irrespective of the value of gravity of model. For the earth, its mean density, $\bar{\rho} = 5.52 \times 10^3\,\mathrm{kg\,m^{-3}}$, gives $\rho_s = 3.68 \times 10^3\,\mathrm{kg\,m^{-3}}$. The actual density of the mantle at the mantle-crust interface is $\rho_s = 3.77 \times 10^3\,\mathrm{kg\,m^{-3}}$. The density distribution of the model in the mantle is compared to the density of the earth's mantle in Fig. 2.

This density distribution can also be derived in a different, illuminating fashion. The locations of the extrema of gravity in a spherically symmetric mass distribution are given by the zeroes of

$$\frac{dg(r)}{dr} = 4\pi G[\rho(r) - (2/3)\,\bar{\rho}(r)] \quad , \tag{4}$$

where $\bar{\rho}(r) \equiv 3M(r)/(4\pi r^3)$ is the mean density interior to r. In a region of constant gravity, $\rho(r) = (2/3)\bar{\rho}(r)$ holds everywhere. But the middle equality of Eq. 2 determines the local mean interior density, $\bar{\rho}(r) = (R/r)\bar{\rho}$, resulting again in Eq. 3.

The derivations of Eqs. 3 and 4, assigned as problems, are pedagogically very useful for the student. They learn and know very well how to differentiate any $f(x)$ with respect to x. It requires practice, however, until the student can write down an expression for a physical quantity like $g(r)$ and perform some mathematical operation on it, for example, differentiate it to find its maximum, automatically.

In the well known problem of a constant gravity, isothermal atmosphere (Walker, 1977), it is the isothermal assumption and the functional form of the ideal gas equation of state that determine the exponential density and pressure distributions. Gravity is an external parameter which plays no role and only affects the scale height of the distributions.

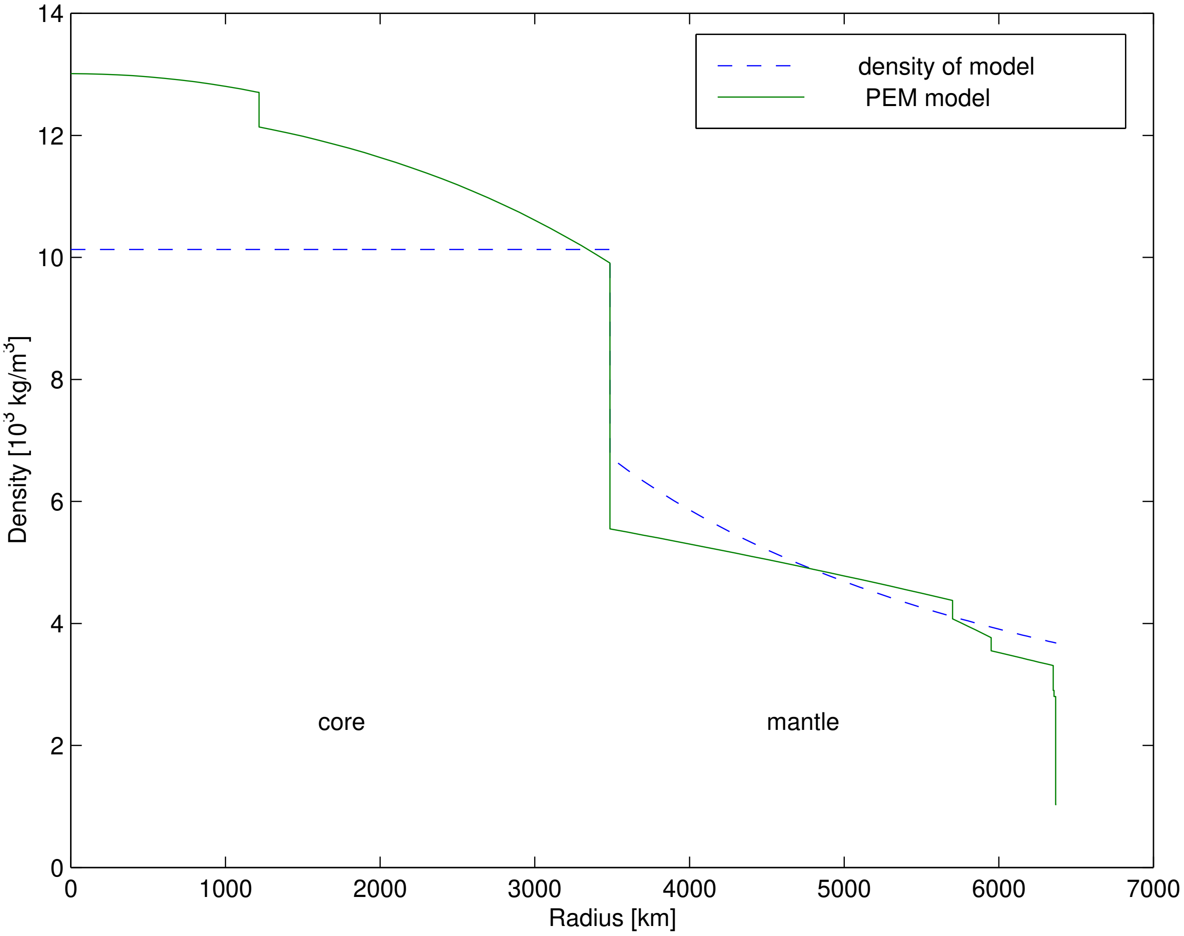

Fig. 2.— The density distribution of the parametric earth model, (PEM), drawn as a solid line, is compared to the density in our model, drawn as a dashed line. The mantle and core, that are major divisions of the earth, are shown.

3. Masses of mantle and core

The mass distribution $M_m(r)$ in the mantle is given directly by Eq. 2,

$$M_m(r) = M\left(\frac{r}{R}\right)^2 \quad , \tag{5}$$

or alternatively by integration of Eq. 1 for the density distribution of the mantle, Eq. 3. Note that the requirement of constant gravity gives the mass interior to r, since in a spherically symmetric mass distribution the mass exterior to r does not contribute to gravity at r. The mass must increase as r^2 to give constant gravity. The density of the core follows from the requirement that the mass of the mantle and the mass of the constant density core give the total mass. If a is the radius of the core, the condition is

$$M_m + M_c = M\left(1 - \frac{a^2}{R^2}\right) + \frac{4\pi}{3}a^3\rho_c = M \quad . \tag{6}$$

Simple algebra gives

$$\rho_c = \left(\frac{R}{a}\right)\bar{\rho} \quad . \tag{7}$$

With $a/R = 0.545$ for the earth, $\rho_c = 10.13 \times 10^3\,\mathrm{kg\,m^{-3}}$.

The discontinuity in density at the core-mantle boundary is a factor 3/2, independent of the mean density of the earth or the position of the core-mantle boundary. In the earth, the actual discontinuity is a factor of 1.80. From Eq. 3, the density of the mantle at the interface is $\rho(a) = 6.75 \times 10^3\,\mathrm{kg\,m^{-3}}$.

The total mass in the core can be written as

$$M_c = M_m(a) = M\left(\frac{a^2}{R^2}\right) \quad . \tag{8}$$

Then $M_c = (4\pi/3)a^3\rho_c$ gives Eq. 7 again. For $a = 0.545R$, $M_c/M = 0.30$, $M_m/M_c = 2.37$. The actual values for the earth are $M_c/M = 0.32$ and $M_m/M_c = 2.13$.

4. Pressure distribution

The pressure distribution in the mantle follows from the equation of hydrostatic equilibrium,

$$\frac{dP(r)}{dr} = -g(r)\rho(r) \quad , \tag{9}$$

and the density distribution of Eq. 3. Since $P(R) = 0$, integration from the surface gives

$$P(r) = \frac{2}{3}\bar{\rho}g_0R\ln\left(\frac{R}{r}\right) = \frac{1}{2\pi}\frac{GM^2}{R^4}\ln\left(\frac{R}{r}\right) = \frac{g_0^2}{4\pi G}\ln\left(\frac{M}{M(r)}\right) \quad . \tag{10}$$

The last equality follows from Eq. 5. It can be derived directly from equations of hydrostatic equilibrium (Mitalas, 1995).

The pressure distribution in the constant density core is the well known

$$P(r) = P_c - \frac{2}{3}\pi G{\rho_c}^2r^2 \quad . \tag{11}$$

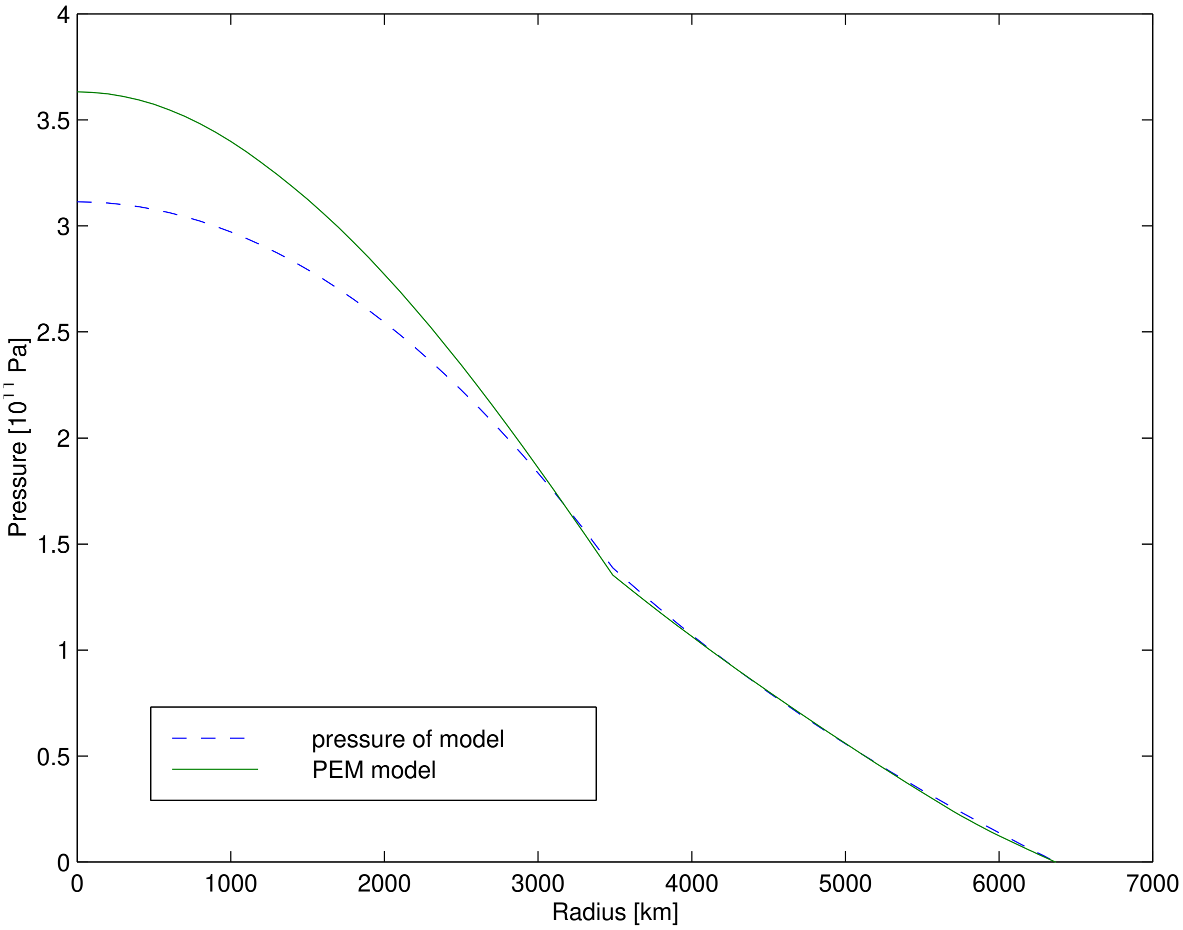

Fig. 3.— The pressure distribution of the parametric earth model (PEM), drawn as a solid line, is compared with the pressure of our model consisting of a constant gravity mantle and constant density core, drawn as a dashed line. The difference of pressure at the center is 13.6%.

The continuity of pressure at the boundary between the core and mantle gives the central pressure,

$$P_c = \frac{2}{3}\pi G\, {\rho_c}^2 a^2 + \frac{2}{3}\bar{\rho} g_0 R \ln\left(\frac{R}{a}\right) = \frac{3}{8\pi}\frac{GM^2}{R^4}\left[1 + \frac{4}{3}\ln\left(\frac{R}{a}\right)\right] \quad . \tag{12}$$

The central pressure is the central pressure of a constant density sphere of radius a and density ρ_c, since $M/R^2 = M_c/a^2$, plus the weight of the overlying mantle from Eq. 10. The pressure distribution in the core is

$$P(r) = \frac{3}{8\pi}\frac{GM^2}{R^4}\left[\left(1 + \frac{4}{3}\ln\left(\frac{R}{a}\right)\right) - \left(\frac{r}{a}\right)^2\right] \quad . \tag{13}$$

The logarithmic term is the pressure of the mantle at the edge of the core. This pressure distribution is compared with the distribution from PEM in Fig. 3. The agreement is excellent in the mantle. The central pressure of the model is $P_c = 3.12 \times 10^{11}$Pa compared to $P_c = 3.61 \times 10^{11}$Pa for the earth, smaller by 13.6%. Since pressure at r is the weight of overlying layers in a truncated cone of unit area at r, the density and gravity exterior to r determine the pressure at r. In the crust, the higher density of the model compensates for the slightly lower gravity; near the core, the lower gravity and the higher density compensate again to give excellent agreement in the mantle. In the core, however, the difference between the pressure of the PEM model and our model increases steadily, since both the gravity and density of the model in the core are lower than those of PEM.

5. Moments of inertia

In spherical polar coordinates, the moment of inertia of the mantle about an axis through the center is

$$I_m = \int_M r^2 sin^2\theta\rho(r)dV = 2\int_0^{2\pi} d\phi \int_0^{\pi/2} \sin^3\theta d\theta \int_a^R r^4\rho(r)dr \quad , \tag{14}$$

where dV is the volume element. Then the density of Eq. 3 gives

$$I_m = \frac{1}{3}MR^2[\,1 - (\frac{a}{R})^4\,] \quad . \tag{15}$$

For the core, the moment of inertia is the well known $I_c = (2/5)M_c a^2$. Hence the moment of inertia of the earth is

$$I = \frac{1}{3}MR^2\left[1 + \frac{1}{5}(\frac{a}{R})^4\right] \quad . \tag{16}$$

With $a/R = 0.545$, $I = 0.3392MR^2$; the observed value is $I = 0.3308MR^2$ (Stacey, 1976).

The moment of inertia of the model can vary from $I = 1/3(MR^2)$, when the model is all mantle, to $I = 2/5(MR^2)$, when the model is all core. The difference is small: $1/(15)MR^2$. Since the observed value of the moment of inertia of the earth, $I = 0.3308MR^2$, is less than one third, the moment of inertia cannot be used to determine a/R. The moment of inertia of the constant gravity mantle is too high. One can see the importance of mass in the outer layers as contributors to the moment of inertia from Fig. 2. Draw a line at $\bar{\rho} = 5.52 \times 10^3\,\mathrm{kg\,m^{-3}}$ to represent a constant density earth, which will have $I = (2/5)MR^2$. Compare with a model which is all mantle, by extending the dashed line as (r^{-1}) to the origin; the moment of inertia will be $I = (1/3)MR^2$. The smaller density in the outer layers compensates for the increase of the density in the inner layers, to infinity at $r = 0$, but still reduces the coefficient of MR^2 from 2/5 to 1/3.

The ratio of moments of inertia of mantle and core is

$$\frac{I_m}{I_c} = \frac{5}{6}\left[(\frac{R}{a})^4 - 1\right] \quad , \tag{17}$$

giving $I_m/I_c = 8.61$. For the earth, the ratio is 7.74. It can be used to determine a/R, giving $a/R = 0.5584$ and $I = 0.3657MR^2$.

6. Elastic Properties of the Mantle

The speeds of elastic waves are determined by the adiabatic bulk modulus K_S and the rigidity μ as well as the density distribution:

$$V_p^2 = \frac{K_S + 4/3\mu}{\rho}, \quad V_s^2 = \frac{\mu}{\rho} \quad , \tag{18}$$

where V_p and V_s refer to longitudinal and transverse waves (Stacey, 1976). The bulk modulus in the constant gravity mantle follows from Eqs. 3 and 10,

$$K_S \equiv -V\frac{\partial P}{\partial V} = \rho\frac{dP}{d\rho} = \frac{2}{3}\bar{\rho}g_0R = \frac{1}{2\pi}\frac{GM^2}{R^4} \quad . \tag{19}$$

Hence the bulk modulus is constant, $K_S = 2.30 \times 10^{11}$Pa. In the mantle of earth, it increases almost linearly with pressure from $K_S = 2.16 \times 10^{11}$Pa at surface to 6.35×10^{11}Pa at the core-mantle boundary. Because the density is constant in the core, $1/K_S = 0$ there. Hence the bulk modulus is infinite in the core. Physically, K_S can be interpreted in terms of properties of a constant density sphere of mass M and radius R:

K_S equals to four thirds the central pressure or ten ninths of the gravitational potential energy per unit volume of such a sphere.

The rigidity of the mantle is determined by the elastic properties of the solid material of the mantle. To be able to estimate wave speeds in the mantle, we assume the approximately valid Poisson relation (Jeffreys, 1976), $K_S = (5/3)\mu$. Then the wave speeds become

$$V_p^2 = \frac{9}{5}\left(\frac{K_S}{\rho}\right) \quad , \qquad V_s^2 = \frac{3}{5}\left(\frac{K_S}{\rho}\right) \quad . \tag{20}$$

Since $\rho V_p^2/2$ is the kinetic energy per unit volume associated with wave motion, it is instructive to calculate the kinetic energy of wave motion in the whole model. The bulk modulus from Eq. 19 gives

$$(\frac{4}{3}\pi R^3)(\frac{1}{2}\rho V_p^2(r)) = \frac{3}{5}\frac{GM^2}{R} = \frac{3}{5}Mg_0R \quad . \tag{21}$$

The physical interpretation of this equation is that the kinetic energy per unit volume of the longitudinal wave equals the gravitational potential energy, per unit volume, of a constant density sphere of mass M and radius R; or three fifths of the work done to move a mass M against gravity g_0 a distance R. The density distribution from Eq. 3 then gives

$$V_p^2(r) = 3V_s^2(r) = (\frac{9}{5}g_0 R)(\frac{r}{R}) = 1.125 \times 10^8(\frac{r}{R})\ \mathrm{m^2 s^{-2}} \quad . \tag{22}$$

Hence the speeds decrease with depth $\propto r^{1/2}$, contrary to the observed increase. In the constant gravity mantle the bulk modulus and rigidity are constant, so any increase of density with depth will give a decrease of wave speeds with depth. Hence the ray paths will curve inward toward the center. In the mantle of the earth, the bulk modulus and rigidity, as well as the density, increase with depth. However, the bulk modulus and rigidity increase with depth more rapidly than the density, resulting in the increase of wave speeds with depth.

The wave speeds given by Eq. 22 have the right order of magnitude, $V_p = 10.6\,\mathrm{km\,s^{-1}}$ at the surface compared to $9.69\,\mathrm{km\,s^{-1}}$ for the earth; at the core-mantle boundary Eq. 22 gives $V_p = 7.9\,\mathrm{km\,s^{-1}}$ compared to $13.8\,\mathrm{km\,s^{-1}}$ for the earth.

Since the wave speeds decrease in the constant gravity mantle, unlike the mantle of earth, we did not consider the propagation of waves and their travel times to illustrate the basic ideas of seismology. However, it is interesting to compare the radial travel time of a longitudinal wave through the mantle with the travel time of particle, starting from rest, and falling in a radial tunnel through the mantle. Does the wave or the particle reach the boundary first? The travel time of the wave is

$$T_w = -\int_R^a \frac{dr}{V_p(r)} = 2\left(\frac{5R}{9g_0}\right)^{1/2} [\,1 - (\frac{a}{R})^{1/2}] \quad . \tag{23}$$

To order of magnitude, this is just free fall under constant gravity, $(1/2)g_0 T_w^2 \approx R$. If the particle starts from rest, the travel time is

$$T_0 = [2(R-a)/g_0]^{1/2} \quad . \tag{24}$$

Hence,

$$\frac{T_w}{T_0} = \frac{\sqrt{10}}{3}\,\frac{1-(\frac{a}{R})^{1/2}}{(1-\frac{a}{R})^{1/2}} \quad . \tag{25}$$

For $a = 0.545R$, $T_w = 0.4091T_0$. The wave reaches the boundary first, more than twice as fast, since it starts out with its initial speed, while the particle starts from rest. The travel time of the particle is 12.8 min. At position of the core boundary, the speed of the particle is almost the speed of the wave there, $V_0 = 0.963V_p = 7.54\,\mathrm{km\,s^{-1}}$. In the limit of $a/R \to 0$, the wave takes longer to reach the center.

The elastic ratio is an observationally determined quantity used to construct the density distribution of the earth. From Eqs. 18 and 19 the elastic ratio ϕ in the mantle is

$$\phi \equiv V_p^2 - \frac{4}{3}V_s^2 = \frac{K_s}{\rho} = g_0 r = 6.25 \times 10^7 (\frac{r}{R})\ \mathrm{m^2\,s^{-2}} \quad . \tag{26}$$

The elastic ratio decreases in the constant gravity mantle, while in the earth it increases. At $r = 5971$km, the crust-mantle boundary, $g_0 r = 5.84 \times 10^7 \mathrm{m^2\,s^{-2}}$, while in the earth $\phi = 5.72 \times 10^7 \mathrm{m^2\,s^{-2}}$ there.

7. Summary

We have presented a model of the earth consisting of a constant density core and a mantle in which the magnitude of gravity is constant. The justification of the assumption of constant gravity in the mantle is the small, determined variation shown in Fig. 1. The assumption of constant gravity gives an $1/r$ density distribution in the mantle and determines the relative proportions of mantle and core masses in terms of the fractional radius of the core. Total mass then determines the density of the core in terms of the fractional radius of the core. In the model, the surface density is 2/3 of the mean density independent of the surface gravity. The discontinuity in density at the core-mantle interface is a factor 3/2, independent of the mean density or gravity of the model or the position of the core-mantle boundary. With the core-mantle boundary fixed at $a/R = 0.545$, the mass of the earth gives $\rho_c = 10.13 \times 10^3\,\mathrm{kg\,m^{-3}}$ for the density of the constant density core and $I = 0.3392MR^2$ for the moment of inertia of the earth compared to the observed value is $I = 0.3308MR^2$. The resulting pressure distribution in the mantle agrees very well with the pressure in the mantle of the earth; in the core the difference increases to a maximum difference of 13.6% at the center. The discontinuity in density at the core-mantle interface of the earth is a factor 1.80, different from the 3/2 in the model.

The observed value of the moment of inertia of the earth, $I = 0.3308MR^2$, cannot be accommodated by the model. The moment of inertia of the constant gravity mantle is too high. However, the ratio of the moments of inertia of mantle and core can be used to determine the location of the core-mantle boundary.

The speeds of elastic waves decrease with depth in the constant gravity mantle, contrary to the increase with depth in the earth, because the bulk modulus is constant and the density is an inverse function of the radius in the mantle. In the earth, wave speeds increase with depth only because the bulk modulus increases inward faster the density. The magnitude of wave speeds has the correct order of magnitude in the mantle.

Although the approximate constancy of gravity in the mantle had been noticed a long time ago (attributed to Benfield in Jeffreys, 1929), the fact that the gravity in the earth's mantle is approximately constant is a surprise to most physicists and astronomers and some geophysicists. There does not appear to be any general physical principle, for example, minimization of total energy including elastic, or some variational principle that seems to demand it. Our model is a useful pedagogical tool to illustrate the principles of hydrostatic equilibrium and acquaint students with the earth. Parts of the model can be presented in lectures or assigned as problems to suit the class.

8. Acknowledgments

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9. References

Bolt, B. A., 1991, The precision of density estimation deep in the earth: Quarterly Journal Royal Astronomical Society, v. 32, p. 367-388.

Brush, S. G., 1980, Discovery of the earth's core: American Journal of Physics, v. 48, p. 705-724.

Bullen, K. E., 1975, The Earth's Density: London, England, Chapman and Hall, 420 p.

Bullen, K. E. and Bolt, B. A., 1985, An introduction to the theory of seismology: Cambridge, England, Cambridge University Press, p. 89.

Dziewonski, A. M., Hales, A. L. and Lapwood, E. R., 1975, Parametrically simple earth models consistent with geophysical data: Physics of Earth and Planetary Interiors, v. 10, p. 12-48, PEM.

Gutenberg, B., 1914, Über Erdbebenwellen. VIIA. Beobachtungen an Registrierungen von Fernbeben in Göttingen und Folgerungen über die Konstitution des Erdkörpers: Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse

p. 1-52 and p. 125-176.
Jeffreys, H., 1924, The Earth: Cambridge, England, Cambridge University Press, 1st ed., 278 p.
Jeffreys, H., 1929, The Earth: Cambridge, England, Cambridge University Press, 2nd ed., 346 p.
Jeffreys, H., 1976, The Earth: Cambridge, England, Cambridge University Press, Cambridge, 6th ed., 574 p.
Mitalas, R., 1995, Limits to pressure in stars and velocity dispersions in stellar systems: Astronomy and Astrophysics, v. 302, p. 924-930.
Schubert, G. and Spohn T., 1990, Thermal history of Mars: Journal of Geophysical Research, v. 95, p. 14,095-14,104.
Snyder, R., 1986, Two-density model of the Earth: American Journal of Physics, v. 54, p. 511-513.
Stacey, Frank D., 1976, Physics of the Earth: New York, John Wiley and Sons, 324 p.
Walker, James C. G., 1977, Evolution of the atmosphere: Macmillan Publishing Co., New York, p3.
Williamson, E. D. and Adams, L. H., 1923, Density distribution in the earth: Journal Washington Academy of Sciences, v. 13, p. 413-428.
Wiechert, E., 1897, Über die Massenverteilung in Innern Erde: Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, p. 221-243.

10. About the authors

Romas Mitalas is professor emeritus in the department of Physics and Astronomy. His research interests are in the astrophysics of stellar structure and evolution. Carol Millar is a Ph. D. student in the department. Her thesis research is on stellar winds from massive, hot stars. This article is the outgrowth of a peripheral problem assigned in a course on stellar structure and evolution.